ULTRAHIGH FREQUENCY TRANSMISSION *and* RADIATION

ULTRAHIGH FREQUENCY TRANSMISSION *and* RADIATION

Nathan Marchand
Lecturer in Electrical Engineering
Columbia University

1947
New York • JOHN WILEY & SONS, Inc.
London • CHAPMAN & HALL, Limited

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

Many were the prophecies that the end of World War II would mark the end of the epoch of ultrahigh frequencies. How wrong they were can be perceived by a glance at the current periodical literature. Radar and navigation systems developed for wartime use and employing many new principles of ultrahigh frequency construction were taken over bodily by civilian users immediately upon removal from confidential classification. Among the many applications are mobile and relay communication systems; frequency modulation systems; mobile, relay, and color television systems; pulse time modulation systems; and many other specialized applications too numerous to mention. The widespread use of ultrahigh frequencies has presented many new problems in transmission and radiation which can be solved only by those familiar with the fundamentals.

I have prepared this book keeping in mind the various applications and have designed it to meet the requirements of both the college student who will study it as a textbook in a prepared course and the practicing engineer who will read it for self-instruction.

As a college textbook, it takes the student step by step through the intricacies of transmission lines, wave propagation, radiation, antennas and antenna arrays, and finally through complex transmission line network analysis which promotes a kind of thinking that is all too often evaded or ignored. The preliminary draft was used as a basis for courses in transmission and radiation at Columbia University, enabling me to correct, modify, and round out the contents. Numerous problems from the course are included for home assignment.

When the book is read for self-instruction, the many clarifying examples and detailed explanations should be helpful. Antennas are not dealt with in a recondite manner but rather as a section of apparatus that has to be designed, constructed, and used. I have drawn on practical experience and the experiences of my associates for many of the explanations and procedures given in the text.

The material for the book was gathered when I was requested to plan and teach an ESMWT course at Columbia University. It was first correlated as a set of notes published at the University in mimeographed form and used for the course. These notes were very kindly received and I was urged by my associates both at the University and at the Federal Telecommunications Laboratories to expand the notes into a book. In doing so I have included pertinent material which has been released for publication by the Army and Navy.

I am indebted to Mr. Andrew Alford, consulting engineer, and also to Mr. H. Busignies, director of the Federal Telecommunications Laboratories, for the many hours spent with each of them in conference and discussion which helped clarify, in my own mind, certain of the fundamental ideas. I am also grateful to Mr. H. T. Kohlhass, editor of *Electrical Communication*, for his kind encouragement and to Mr. H. H. Buttner, president of the Federal Telecom-

munications Laboratories, for making drafting and clerical help available to me. I wish to express my sincere thanks to Professor John Ragazzini of Columbia University for his detailed criticism and to Mr. Lewis Winner, editor of *Communications,* for his encouragement and editorial help. I am also deeply indebted to my wife Ernesta, for her encouragement, patience, and help.

NATHAN MARCHAND

NEW YORK CITY
May, 1947

CONTENTS

Chapter 1

TRANSMISSION LINES

1·1 INTRODUCTION

Transmission lines were first used to transmit power from one point to another. This still remains one of the primary applications, but with the advent of higher radio frequencies transmission lines have been adapted to many other uses. They are employed now as circuit elements, transformers, measuring devices, and attenuators. To understand the foregoing applications it is necessary to study the fundamental transmission line equations.

It is assumed that the transmission lines to be considered are so spaced and constructed that circuit theory will be applicable. The spacing between lines has to be small compared to the wavelength of the energy being transmitted. If such is not the case then circuit theory in its common form will not apply; the transmission line will have to be analyzed as a wave guide.

1·2 COAXIAL AND BALANCED TRANSMISSION LINES

The two most common types of transmission lines are the coaxial and the balanced transmission lines. The balanced transmission line is a parallel wire line consisting of two parallel conductors carrying a balanced voltage to ground. The energy is conducted on the outside of the lines by waves traveling through the space surrounding the lines and is thus guided by the lines. Because of skin effect the current flow is usually confined to the surface of the wires. In fact, as the wires are spaced closer together the proximity effect of one current on the other causes the currents to be further confined to the adjacent surfaces. To maintain uniformity on balanced lines, thus avoiding unbalanced conditions to ground, and to prevent radiation, a shield is usually employed to surround the two wires. A cross section of a shielded balanced transmission line is shown in Figure 1·1.

The coaxial line, the other type of common transmission line, consists of a wire located in the center of the tube, as shown in Figure 1·2. In this case the energy is transmitted in the space between the outside of the inner conductor and the inside of the outer conductor. The voltage is impressed between the inner conductor and the shield, or outer con-

ductor. The shield is usually grounded at both the receiving and the sending end so that this type of line is often referred to as an unbalanced transmission line.

In both types of transmission line, if the shields are continuous, there is no loss of energy by radiation.

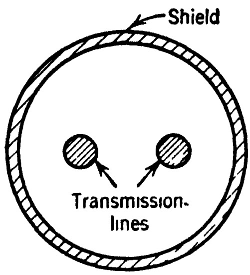


FIG. 1·1 A cross section of a shielded balanced transmission line.

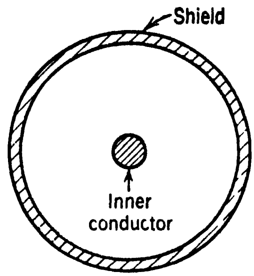


FIG 1·2 A cross section of a coaxial transmission line.

1·3 DISTRIBUTED CONSTANTS

In ordinary circuit theory it is assumed that all constants are lumped constants. It is thus implied that in a two-terminal network a current starts to flow when a voltage is impressed across the two terminals, the current being related to the impressed voltage by the constant. However, let us consider the case wherein a number of resistors, condensers, and inductances are connected in series. Here there exist what are known as transients. It takes a specific time interval for any change in voltage to be felt throughout the circuit. In alternating current theory these transients also exist, and a somewhat similar effect is present in the steady state condition. This latter effect is not known as a transient but rather as a phase shift.

Now let us consider a transmission line. Each unit length possesses its own inductance, capacitance, series resistance, and leakage conductance. Thus each unit length is a small network wherein the effect of a change in voltage at its input will take a specific time to be felt at its output. In this manner the effect of a change in voltage at the input to the transmission line will travel down the line at a speed dependent on the values of the line constants. The resultant waves are known as traveling waves. It must be remembered that all constants are really distributed but for engineering accuracy may be assumed to be lumped when the error is negligible. Similarly, if the transmission line is very small with respect to a wavelength, the constants can also be

assumed to be lumped. This postulation will become clear after the equations for transmission lines have been derived.

1·4 Derivation of the transmission line equations

Let us consider a transmission line with the following constants:

R—series resistance per unit length in ohms;
L—series inductance per unit length in henrys;
G—shunt conductance per unit length in mhos;
C—shunt capacitance per unit length in farads.

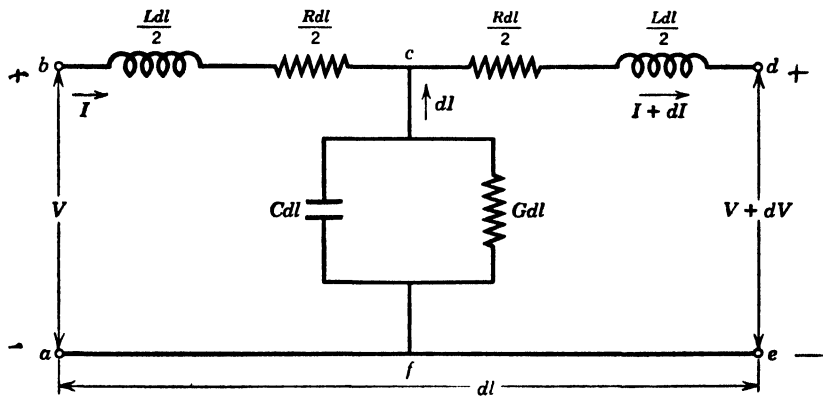


Fig. 1·3 The equivalent circuit for a differential length of transmission line.

These are the fundamental constants of a transmission line. As distinguished from G, R is the resistive component of the series impedance causing a drop in the voltage V, whereas G is the resistive component of the shunt admittance causing a drop in the current I. The equivalent circuit of a transmission line differential length, dl, is shown in Figure 1·3. The constants for the circuit shown, since a length dl is assumed, are the foregoing unit values times the differential length. The equivalent T network is illustrated in the figure where the distributed values are shown as lumped constants. The series element consists of two branches each having a resistance of $\frac{R\,dl}{2}$ ohms and an inductance of $\frac{L\,dl}{2}$ henrys. The shunt element is a parallel circuit consisting of a capacitance of $C\,dl$ farads and a conductance of $G\,dl$ mhos. To obtain the steady state solution, a constant frequency voltage V, at a frequency f, is assumed to be impressed across the input terminals ab.

This will cause a current I to flow in the line. A differential current, dI, will flow across the shunt impedance and, since all signs will be assumed to be positive until shown to be otherwise, it is added to the current I. Hence I plus dI will flow in the branch cd from c to d. Similarly the voltage across ed will differ from the voltage input across ab, a difference noted as dV. Again, the positive sign is used and the correct sign will be indicated by the solution.

The circuit equation around the network $abdea$ is given by

$$V - I\left(j\omega\frac{L\,dl}{2} + \frac{R\,dl}{2}\right)$$

$$- (I + dI)\left(j\omega\frac{L\,dl}{2} + \frac{R\,dl}{2}\right) - (V + dV) = 0 \qquad (1\cdot1)$$

where

$$j^2 = -1,$$

$$\omega = 2\pi f.$$

Equation 1·1 is simplified by carrying out the multiplications indicated and neglecting second-order differentials. It yields for the change in voltage

$$dV = -I(R + j\omega L)\,dl \qquad (1\cdot2)$$

The change in voltage has a minus sign preceding it, indicating that the voltage change, dV, is negative if the current, I, is positive and is positive if the current, I, is negative.

The voltage across cf, V_{fc}, is obtained by subtracting the voltage across the branch bc from the input voltage V:

$$V_{fc} = V - I\left(\frac{R\,dl}{2} + j\omega\frac{L\,dl}{2}\right) \qquad (1\cdot3)$$

The second term of this equation, however, is equal to $dV/2$, as shown in Equation 1·2. Making this substitution, we obtain

$$V_{fc} = V + \frac{dV}{2} \qquad (1\cdot4)$$

To obtain dI, V_{fc} is multiplied by the shunt admittance of the branch cf. This shunt admittance consists of a parallel capacitance and resistance, as shown in Figure 1·3. The voltage across fc, from f to c, is assumed positive. Since the current dI is also shown as positive from f to c, multiplying the voltage V_{fc} by the admittance will yield $-dI$:

$$-dI = \left(V + \frac{dV}{2}\right)(G\,dl + j\omega C\,dl) \qquad (1\cdot5)$$

Equation 1·5 is simplified by carrying out the multiplication indicated and neglecting the second-order differentials:

$$dI = -V(G + j\omega C)\,dl \tag{1·6}$$

We note how similar Equation 1·6 is to the voltage variation Equation 1·2.

To simplify calculations, Z is substituted for the series impedance and Y for the shunt admittance:

$$\begin{aligned} Z &= R + j\omega L \\ Y &= G + j\omega C \end{aligned} \tag{1·7}$$

Substituting Equations 1·7 into Equations 1·2 and 1·6 and dividing through by the differential length dl, we obtain

$$\begin{aligned} \frac{dV}{dl} &= -ZI \\ \frac{dI}{dl} &= -YV \end{aligned} \tag{1·8}$$

To obtain similar equations for V and I, each equation of Equations 1·8 is differentiated once with respect to l:

$$\begin{aligned} \frac{d^2V}{dl^2} &= -Z\frac{dI}{dl} \\ \frac{d^2I}{dl^2} &= -Y\frac{dV}{dl} \end{aligned} \tag{1·9}$$

The values of dI/dl and dV/dl from Equations 1·8 are substituted into Equations 1·9 yielding

$$\begin{aligned} \frac{d^2V}{dl^2} &= YZV \\ \frac{d^2I}{dl^2} &= YZI \end{aligned} \tag{1·10}$$

The two equations of Equations 1·10 are each functions of one variable and state that the second derivative of this variable is a constant times the variable. They indicate that the salutions for I and V will be similar in form; the method of solution applied to one will be directly applicable to the other. An exponential function satisfies Equations 1·10. Treating first the voltage equation, we assume that

$$V = ke^{\Gamma l} \tag{1·11}$$

The symbol k is an arbitrary constant; e, the base of the natural logarithms (2.71828); and Γ, the propagation constant, so called because it determines the variation of V with the length l. Taking the second derivative of V (as given by Equation 1·11) with respect to l, we obtain

$$\frac{d^2V}{dl^2} = \Gamma^2 k e^{\Gamma l} \tag{1·12}$$

Equations 1·11 and 1·12 can now be substituted into the first part of Equations 1·10:

$$\Gamma^2 k e^{\Gamma l} = YZke^{\Gamma l} \tag{1·13}$$

The factor $ke^{\Gamma l}$ appears on both sides of Equation 1·13. It may be factored out and the solution for Γ^2 obtained:

$$\Gamma^2 = YZ \tag{1·14}$$

Taking the square root of both sides of Equation 1·14, we obtain the value of the propagation constant Γ in terms of Y and Z, the original transmission line constants:

$$\Gamma = \pm\sqrt{YZ} \tag{1·15}$$

This equation shows that the complete solution for V will consist of two parts, one where Γ has a positive sign and the other where Γ has a negative sign. To avoid confusion, henceforth the symbol Γ is employed to represent the absolute value of $\sqrt{ZY}$ and the plus and minus values necessary for the complete solution are obtained by the use of signs. In an equation like Equation 1·15, where the exponent has two signs, the result consists of two exponentials, each with its own constant. Thus the complete expression for V will be

$$V = k_1 e^{\Gamma l} + k_2 e^{-\Gamma l} \tag{1·16}$$

The letters k_1 and k_2 are arbitrary constants whose values will be determined by the length of line, the termination, and the source.

Equations 1·10 show that the equation for I is similar to the equation for V. Therefore, the solution for I can be obtained immediately; it will differ from Equation 1·16 only by the constants involved:

$$I = k_3 e^{\Gamma l} + k_4 e^{-\Gamma l} \tag{1·17}$$

Thus both the current, I, and the voltage, V, have the same propagation constant. The letters k_3 and k_4 are arbitrary constants analogous to k_1 and k_2.

1·5 The transmission line equation constants

The values of the constants, k_1, k_2, k_3, and k_4 are obtained by substituting the solutions for V and I, as given in Equations 1·16 and 1·17, into the original continuity equations, Equations 1·8. Since these equations involve the first derivative, the first derivatives of Equations 1·16 and 1·17 with respect to l are obtained:

$$\begin{aligned} \frac{dV}{dl} &= \Gamma k_1 e^{\Gamma l} - \Gamma k_2 e^{-\Gamma l} \\ \frac{dI}{dl} &= \Gamma k_3 e^{\Gamma l} - \Gamma k_4 e^{-\Gamma l} \end{aligned} \tag{1·18}$$

Equations 1·16, 1·17, and 1·18 are now substituted into Equations 1·8:

$$\begin{aligned} \Gamma k_1 e^{\Gamma l} - \Gamma k_2 e^{-\Gamma l} &= -Zk_3 e^{\Gamma l} - Zk_4 e^{-\Gamma l} \\ \Gamma k_3 e^{\Gamma l} - \Gamma k_4 e^{-\Gamma l} &= -Yk_1 e^{\Gamma l} - Yk_2 e^{-\Gamma l} \end{aligned} \tag{1·19}$$

Equations 1·19 can be simplified by factoring out similar exponential terms:

$$\begin{aligned} 0 &= (Zk_3 + \Gamma k_1)e^{\Gamma l} + (Zk_4 - \Gamma k_2)e^{-\Gamma l} \\ 0 &= (Yk_1 + \Gamma k_3)e^{\Gamma l} + (Yk_2 - \Gamma k_4)e^{-\Gamma l} \end{aligned} \tag{1·20}$$

To solve these two equations for the arbitrary constants, we note that $e^{\Gamma l}$ is an exponential which increases in value with an increase in l, whereas $e^{-\Gamma l}$ is an exponential which decreases in value with an increase in l. However, Equations 1·20 state that two sums of these exponentials, each multiplied by its own factor, are equal to zero. The only solution is for the resultant constants, the multiplying factors of the exponentials, to be individually equal to zero. Thus each factor enclosed in parentheses in Equations 1·20 can be equated to zero:

$$\begin{aligned} Zk_3 + \Gamma k_1 &= 0 \\ Zk_4 - \Gamma k_2 &= 0 \\ Yk_1 + \Gamma k_3 &= 0 \\ Yk_2 - \Gamma k_4 &= 0 \end{aligned} \tag{1·21}$$

Not all the four resultant equations are independent Any two of them involving all four constants are sufficient to secure solutions for

two of the constants in terms of the other two. The solutions obtained are

$$k_1 = -\frac{Z}{\Gamma} k_3 = -\frac{\Gamma}{Y} k_3$$
$$k_2 = \frac{Z}{\Gamma} k_4 = \frac{\Gamma}{Y} k_4 \tag{1·22}$$

It is obvious from Equations 1·22 that Z/Γ and Γ/Y must be equal to the same quantity. Substituting $\sqrt{ZY}$ for Γ, its value as given in Equation 1·15, we find both quantities to be identical and equal to $\sqrt{Z/Y}$. This quantity, the square root of the series impedance divided by the shunt admittance, is very important in transmission line analysis. It has the dimensions of an impedance and is usually called the characteristic impedance. It will be represented by the symbol Z_0. Thus

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \tag{1·23}$$

Z_0, as obtained in Equation 1·23, is substituted into the constants relationships, as given in Equations 1·22 so that

$$k_3 = -\frac{k_1}{Z_0}$$
$$k_4 = \frac{k_2}{Z_0} \tag{1·24}$$

Equations 1·24 show that there are only two independent arbitrary constants in the transmission line equation solutions. They are the amplitude constants and phase relationships which are determined by the boundary conditions of the transmission line being used. Substituting these values into the solutions as given by Equations 1·16 and 1·17, we obtain

$$V = k_1 e^{\Gamma l} + k_2 e^{-\Gamma l}$$
$$I = -\frac{k_1}{Z_0} e^{\Gamma l} + \frac{k_2}{Z_0} e^{-\Gamma l} \tag{1·25}$$

Thus the constants in the current equation are directly related to the constants in the voltage equation by the characteristic impedance of the transmission line. We see, however, that in the current equation the exponential with the positive factor has a negative sign in front of its constant.

1·6 THE WAVE CHARACTER OF THE TRANSMISSION LINE EQUATIONS

A steady state single frequency voltage has been assumed in the foregoing solutions. It means that the voltage and current in the solution are actually sine wave variations. The conventional method of representing a sine wave variation is as the real part of an exponential function with an imaginary exponent:

$$\begin{aligned} V &= \mathcal{R}(V' e^{j\omega t}) \\ I &= \mathcal{R}(I' e^{j\omega t}) \end{aligned} \tag{1·26}$$

where $\mathcal{R}$ means "the real part of" and V' and I' are complex quantities, not necessarily in phase with one another. The magnitudes of V' and I' are the peak amplitudes of the voltage and current. Actually, of course, in alternating current calculations the symbol $\mathcal{R}$ is left out and the provision that only the real part is taken is understood. In Equations 1·25, since V is a voltage, k_1 and k_2 must each have the dimensions of a voltage. Thus

$$\begin{aligned} k_1 &= E^- e^{j\omega t} \\ k_2 &= E^+ e^{j\omega t} \end{aligned} \tag{1·27}$$

where ω is equal to 2π times the frequency, f, of the impressed voltage. The reason for using plus and minus superscripts to distinguish between the two voltage values will become evident when the wave character of the solution is discussed.

Similarly, the constants in the current equation are currents since they are related to the voltage constants by the impedance Z_0. Hence

$$\begin{aligned} -\frac{k_1}{Z_0} &= I^- e^{j\omega t} \\ \frac{k_2}{Z_0} &= I^+ e^{j\omega t} \end{aligned} \tag{1·28}$$

We note that the expression for the current I^+ has the same sign preceding it as the expression for E^+ but that in the case of I^- and E^- opposite signs are obtained. Substituting the relationships given by Equations 1·27 and 1·28 into the transmission line equation solutions, Equations 1·25, and combining the exponential exponents, we obtain

$$\begin{aligned} V &= E^+ e^{j\omega t - \Gamma l} + E^- e^{j\omega t + \Gamma l} \\ I &= I^+ e^{j\omega t - \Gamma l} + I^- e^{j\omega t + \Gamma l} \end{aligned} \tag{1·29}$$

However, the propagation constant, Γ, is also a complex function, so that

$$\Gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta \tag{1·30}$$

α is the real part of Γ and determines the variation in magnitude of V and I; hence, α is called the attenuation constant per unit length. β is the imaginary part of Γ determining the variation of phase, with the length, l, of V and I; hence, β is called the phase constant per unit length. Substituting these values for Γ in Equations 1·29 and separating the real and imaginary components of the exponents of the exponentials, we obtain

$$\begin{aligned} V &= E^+ e^{j(\omega t - \beta l)} e^{-\alpha l} + E^- e^{j(\omega t + \beta l)} e^{\alpha l} \\ I &= I^+ e^{j(\omega t - \beta l)} e^{-\alpha l} + I^- e^{j(\omega t + \beta l)} e^{\alpha l} \end{aligned} \tag{1·31}$$

The relationships between the E and I constants in these equations may be obtained by substituting Equations 1·28 into Equations 1·27:

$$\begin{aligned} I^+ &= \frac{E^+}{Z_0} \\ I^- &= -\frac{E^-}{Z_0} \end{aligned} \tag{1·32}$$

Equations 1·31 are the traveling wave solutions of the transmission line equations. E^+ and I^+ can be shown to be waves traveling in the plus l direction. To analyze, let us consider the phase factor $e^{j(\omega t - \beta l)}$. This factor determines the phase of the current and voltage at any point l, with reference to the point l equal to zero. In Figure 1·4A are plotted two curves of E^+ versus l, one for the time t and the other for the time t_1, which is greater than t. We can see from the figure that the wave has moved in the direction of plus l. The phase factor indicates that if t is increased at any point, l, the instantaneous phase of the voltage will increase by the same amount, since the phase is given by $(\omega t - \beta l)$. Another way of stating it is to say that as ωt is increased, βl has to increase an equal amount to obtain the same instantaneous phase. The attenuation factor $e^{-\alpha l}$ causes the wave to decrease in amplitude as l is increased. $E^+ e^{-\alpha l}$ is also shown in Figure 1·4A as a dashed line curve. This curve acts as an envelope for the wave traveling in the plus l direction.

The E^- and the I^- functions are waves traveling in the minus l direction. Let us consider the phase factor $e^{j(\omega t + \beta l)}$. In this case, if ωt is increased, βl must be decreased to maintain the same phase. This is illustrated in Figure 1·4B, which shows the curves of E^- versus l for

the time t and for the time t_1, where t_1 is greater than t. It can be seen from the figure that the E^- wave is traveling in the minus l direction. Thus we find from Equations 1·31 that any steady state condition on

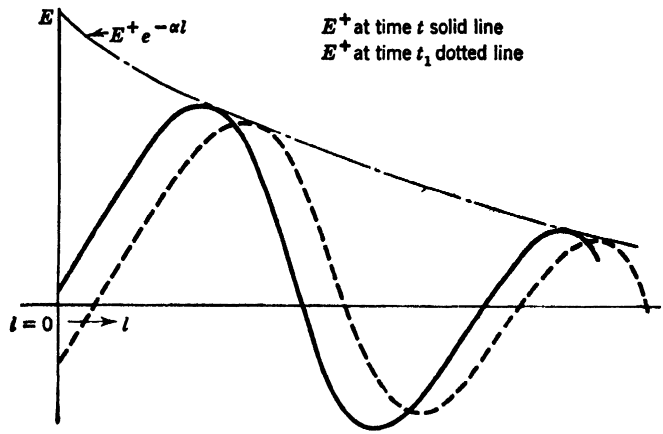


FIG. 1.4*A* Curves of E^+ versus l for the time t and the time t_1, which is greater than t

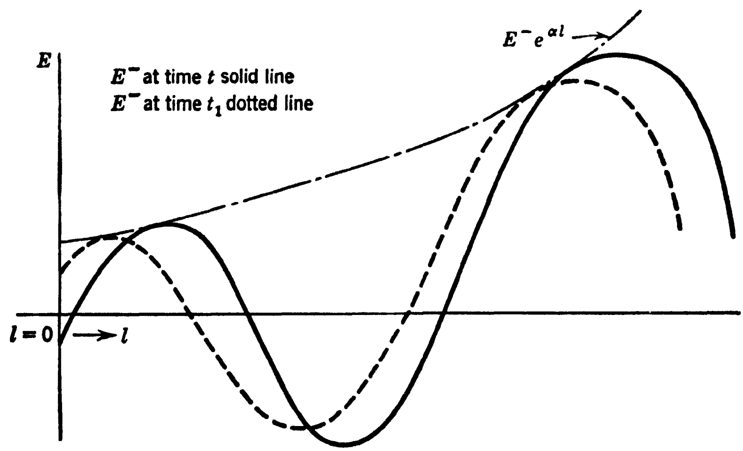


FIG. 1·4*B* Curves of E^- versus l for the time t and the time t_1, which is greater than t. In this case the curve appears to travel in the minus l direction.

a transmission line at a single frequency can be completely represented by the sum of two traveling waves traveling in opposite directions along the transmission line. The end conditions determine the constants,

E^+, E^-, I^+, and I^-. Each of these constants consists of an amplitude and a phase factor.

Inasmuch as these waves are traveling along the transmission line, they will have a certain effective length. To determine the length of line equal to one wavelength, in other words the distance between points of similar phase on the traveling waves, let

$$\beta\lambda = 2\pi \tag{1·33}$$

where λ is the wavelength desired. In the equation β is expressed in radians per unit length. Dividing both sides of Equation 1·33 by β, we get

$$\lambda = \frac{2\pi}{\beta} \tag{1·34}$$

This states that the wavelength along a transmission line is equal to 2π divided by the phase constant, where the phase constant is expressed in radians per unit length.

The waves will also have a definite velocity. The velocity, as in ordinary wave calculations, is equal to the wavelength times the frequency. Calling v the velocity, we find that

$$v = \lambda f \tag{1·35}$$

This is the velocity with which the phase of the wave advances down the line. It is usually called, appropriately, the phase velocity. Substituting for λ from Equation 1·34 and letting ω equal $2\pi f$, we obtain

$$v = \frac{\omega}{\beta} \quad \text{or} \quad \beta = \frac{\omega}{v} \tag{1·36}$$

In ordinary open air transmission lines, this velocity is very nearly equal to the speed of light, 3×10^{10} centimeters per second.

The transmission line equations are sometimes expressed in terms of the velocity of propagation. They may be obtained by substituting Equation 1·36 into Equations 1·31. The ω may be factored out so that

$$\begin{aligned} V &= E^+ e^{j\omega\left(t - \frac{l}{v}\right)} e^{-\alpha l} + E^- e^{j\omega\left(t + \frac{l}{v}\right)} e^{\alpha l} \\ I &= I^+ e^{j\omega\left(t - \frac{l}{v}\right)} e^{-\alpha l} + I^- e^{j\omega\left(t + \frac{l}{v}\right)} e^{\alpha l} \end{aligned} \tag{1·37}$$

Another factor, sometimes used in a transmission line calulations, is the time delay, δ, between putting an impulse on the line and receiving

it further down the line. If the phase velocity v is independent of the frequency, then

$$\delta = \frac{s}{v} \tag{1·38}$$

where s is the length of line and v the velocity of the wave.

1·7 The dissipationless line

In many cases α is so small that we may neglect it completely and still obtain an accurate result. This is true when copper conductor transmission lines are used in an air dielectric at frequencies below 500 megacycles. Of course, under resonant conditions the loss factors cannot be neglected in calculating impedances, but under normal conditions the solutions are quite accurate. Assuming, then, that R and G in the propagation constant are both zero, we obtain for Γ

$$\Gamma = \sqrt{(j\omega L)(j\omega C)} = j\omega\sqrt{LC} \tag{1·39}$$

Γ will now possess only an imaginary term. Thus α is zero, and the expression for the phase factor β becomes

$$\beta = \omega\sqrt{LC} \tag{1·40}$$

Similarly, the expression for the characteristic impedance Z_0 becomes

$$Z_0 = \sqrt{\frac{L}{C}} \tag{1·41}$$

Since, in the dissipationless case, there are no imaginary terms in Z_0, it is always a pure resistance.

The equation for the wavelength in the dissipationless line is, from Equation 1·34,

$$\lambda = \frac{1}{f\sqrt{LC}} \tag{1·42}$$

where the 2π's have been canceled out in the numerator and denominator.

The phase velocity for the dissipationless line is, from Equation 1·36,

$$v = \frac{1}{\sqrt{LC}} \tag{1·43}$$

In a dissipationless line in free space the velocity is equal to 3×10^{10} centimeters per second. It means that the relationship of L to C in a

dissipationless line is always constant, irrespective of the size or type of line

Immersing the line in a dielectric increases the shunt capacitance, C, while all other values remain unchanged The phase constant, β, from Equation 1 40, increases as the square root of the change in C whereas the phase velocity, v, varies inversely as the square root of the change in C Hence the wave will travel more slowly with the line immersed in a dielectric Since the frequency will remain the same, the wavelength λ will decrease

The transmission line equations for the dissipationless case become

$$\begin{aligned} V &= E^{+}e^{j(\omega t-\beta l)} + E^{-}e^{j(\omega t+\beta l)} \\ I &= I^{+}e^{j(\omega t-\beta l)} + I^{-}e^{j(\omega t+\beta l)} \end{aligned} \tag{1 44}$$

We see how the magnitudes of the waves traveling along the lines remain constant, they do not vary with l as when the lines are not dissipationless This does not mean that the plus waves and the minus waves are equal The relationship between those two waves are dependent on the terminating impedance, as will be shown later When a dissipationless transmission line is used, the only element that varies with l, as the wave is propagated down the line, is the phase of the wave.

EXAMPLE 1 1 A voltage, at a frequency of 100 megacycles, impressed on the input terminals of a dissipationless transmission line which has a capacitance of 15 $\mu\mu$f per meter Find the inductance per unit length, the characteristic impedance, the phase constant, and the wavelength along the line

From Equation 1 43, where v is 3×10^{10} centimeters per second, or 3×10^{8} meters per second,

$$L = \frac{1}{C(3 \times 10^{8})^2}$$

$$L = \frac{1}{(15 \times 10^{-12})(9 \times 10^{16})}$$

$$L = 0\ 74\ \mu \text{ henrys per meter} \qquad \textit{Ans. (a)}$$

From Equation 1 41 the characteristic impedance is obtained.

$$Z_0 = \sqrt{\frac{0\ 74 \times 10^{-6}}{15 \times 10^{-12}}}$$

$$Z_0 = 222 \text{ ohms} \qquad \textit{Ans (b)}$$

From Equation 1·40 the phase constant is given by

$$\beta = \frac{\omega}{v}$$

$$\beta = \frac{2\pi(100 \times 10^6)}{3 \times 10^8}$$

$$\beta = 2.09 \text{ radians per meter} \qquad \textit{Ans. } (c)$$

The wavelength from Equation 1·35 may be expressed in terms of the velocity and the frequency.

$$\lambda = \frac{v}{f}$$

$$\lambda = \frac{3 \times 10^8}{100 \times 10^6}$$

$$\lambda = 3 \text{ meters} \qquad \textit{Ans } (d)$$

1·8 Line constants for the dissipationless line

There are only two line constants necessary to determine the behavior of a dissipationless transmission line. They are the series inductance L and the shunt capacity C. Their values may be determined from well-known laws of physics.

For the coaxial transmission line, shown in Figure 1·5*a*, whose shield has an inside diameter of D and whose inner conductor has a diameter of d, the values of L and C are given by

$$L = \frac{k_m \mu_0}{2\pi} \log_e \frac{D}{d} \text{ henrys per meter} \tag{1·45}$$

$$C = \frac{2\pi k_e \epsilon_0}{\log_e \dfrac{D}{d}} \text{ farads per meter} \tag{1·46}$$

The symbols k_e and k_m are, respectively, the relative dielectric and the relative permeability constants of the medium between the conductors. ϵ_0 and μ_0 are, respectively, the dielectric and the permeability constants of free space in M.K.S. units. From the two equations for L and C, the values of v and Z_0 can be obtained directly. As stated in Equation 1·43, the velocity is equal to one over the square root of L times C.

$$v = \frac{1}{\sqrt{k_m k_e \mu_0 \epsilon_0}} \tag{1·47}$$

where ϵ_0 is equal to $(1/36\pi) \times 10^{-9}$ farad per meter and μ_0 is equal to $4\pi \times 10^{-7}$ henry per meter. With air dielectric, k_e and k_m are each

equal to one. Hence the velocity of propagation of an air dielectric transmission line is 3×10^8 meters per second. This checks with the previous statements. Notice that the configuration of the line, which in this case are the dimensions D and d, do not enter into the velocity equation. The velocity is determined purely by the dielectric material surrounding the lines and filling the space between. Inasmuch as there is no field outside the outer shield, the dielectric outside the shield will not influence the constants of the line.

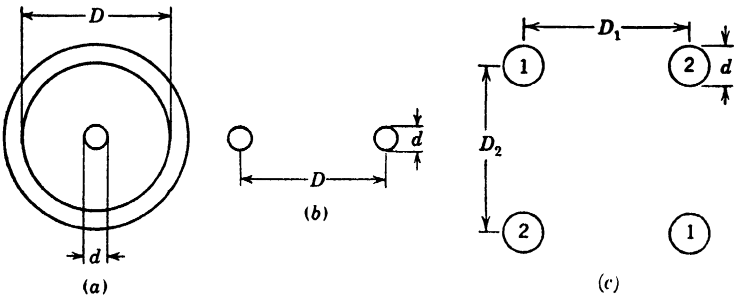


FIG. 1·5 Dimensions of the coaxial, balanced, and four-wire transmission lines as used for obtaining the line constants.

Substituting the values of L and C, as given in Equations 1·45 and 1·46, into the equation for Z_0 and simplifying the result for air dielectric, we obtain

$$Z_0 = 138 \log_{10} \frac{D}{d} \text{ ohms} \tag{1·48}$$

Similarly, the procedure may be repeated for the parallel wire transmission line shown in Figure 1·5*b*; the transmission line wires have a diameter of d units and are spaced a distance D units apart between centers. As long as D and d are in the same units, the characteristic impedance obtained, which is dependent on the ratio, will be correct. The velocity of propagation will again be the velocity of light and the characteristic impedance will be given by

$$Z_0 = 120 \cosh^{-1} \frac{D}{d} \text{ ohms} \tag{1·49}$$

$\text{Cosh}^{-1}(D/d)$ is the value of the angle in radians whose hyperbolic cosine is equal to D/d. The hyperbolic cosine is obtained because the currents are attracted to one another and become more and more confined to the inside adjacent surfaces as the wires are brought closer together.

For large values of D/d this is inconsequential, and the equation becomes

$$Z_0 = 276 \log_{10} \frac{2D}{d} \text{ ohms} \tag{1·50}$$

When the two wires are enclosed in a shield, of diameter ρ, symmetrically placed around the two wires, the characteristic impedance becomes

$$Z_0 = 276 \log_{10} \left\{ \frac{2D}{d} \left[\frac{1 - \left(\frac{D}{\rho}\right)^2}{1 + \left(\frac{D}{\rho}\right)^2} \right] \right\} \text{ ohms} \tag{1·51}$$

The four-wire-transmission line, illustrated in Figure 1·5*c* consists of similar wires, of diameter d, placed at the corners of a rectangle whose dimensions are D_1 and D_2. Wires in diagonally opposite corners carry parallel currents. The characteristic impedance of such a transmission line is given by

$$Z_0 = 138 \log_{10} \frac{2D_2}{d\sqrt{1 + \left(\frac{D_2}{D_1}\right)^2}} \tag{1·52}$$

All the above expressions for the characteristic impedance assume no supporting members. This, of course, cannot be true. If beads are used to support the line, the effective capacitance is increased. It may be taken into account, approximately, by assuming that the line consists of sections of line with an air dielectric in series with sections of line with the bead dielectric. The total capacitance per meter of line length may then be calculated by adding the capacitance of the air line present in one meter of line length to the capitance of the bead dielectric line present in one meter of line length. The inductance with beads is the same as for an air line. These values of capacitance and inductance can now be used to calculate the characteristic impedance.

Example 1·2 Determine the characteristic impedance of a parallel wire transmission line whose ratio of distance, D, between centers of the wire to the diameter, d, of the wires is 8.

The Z_0 may be obtained by substituting into Equation 1·50.

$$Z_0 = 276 \log_{10} 16 = 332 \text{ ohms} \qquad \textit{Ans.}$$

Example 1·3 What is the effect on the Z_0 when the parallel wire line of example 1·2 is enclosed in a shield whose diameter, ρ, is three times the spacing, D, between wire centers. The shield is completely filled with a dielectric which has a k_e of 2.40. It is in effect, then, a parallel wire, solid dielectric, shielded transmission line.

The Z_0 of the line in air is given by Equation 1·51 so that the Z_0 in air dielectric is

$$Z_0\,(\text{air}) = 276 \log_{10}\left\{16\left[\frac{1-\left(\frac{1}{3}\right)^2}{1+\left(\frac{1}{3}\right)^2}\right]\right\}$$

$$Z_0\,(\text{air}) = 276 \log_{10} 12.8$$

$$Z_0\,(\text{air}) = 306 \text{ ohms}$$

However, since the characteristic impedance is equal to the square root of L over C, when the C is increased by a factor of k_e the characteristic impedance is decreased, being inversely proportional to the square root of the dielectric constant. Thus

$$Z_0\,(\text{dielectric}) = \frac{1}{\sqrt{k_e}} Z_0\,(\text{air})$$

$$Z_0 = \frac{1}{\sqrt{2.40}} 306 = 198 \text{ ohms} \qquad \textit{Ans.}$$

1·9 Substitution of boundary conditions in the dissipationless line case

For the traveling wave equations, given in Equations 1·44, to be useful, the values of E^+, E^-, I^+, and I^- must be obtained. They are related to the following transmission line factors: the length of line,

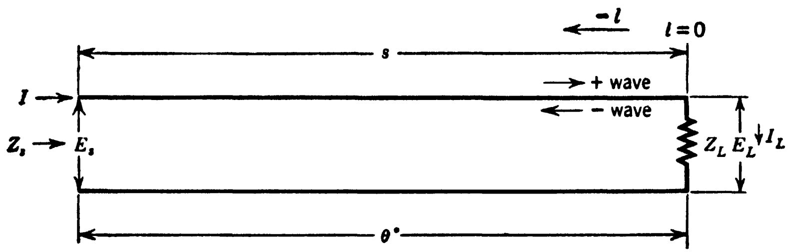


Fig. 1·6 The basic transmission line circuit for a transmission line with a load impedance Z_L across its output

the load conditions, and the generator conditions. They are called the boundary conditions of the equations and may be substituted directly into the equations to obtain the resultant values of the constants. Again, a dissipationless line is assumed.

The basic transmission line circuit is shown in Figure 1·6. The transmission line has a characteristic impedance of Z_0 and a propaga-

tion factor of $j\beta$. The line is s units long and has a load impedance, Z_L, connected across the receiving end. A voltage E_s is impressed across the sending end, directly across the terminals of the transmission line.

For convenience of calculation, assume that l is zero at the receiving end of the line and has a value, at any point on the line, equal to the negative of the distance along the line from the point in question to the receiving end.

At l equal to zero, that is, at the receiving end, the sum of the two traveling waves must equal the voltage across Z_L, namely, E_L. Similarly, the sum of the two current waves must equal the current through Z_L, namely, I_L. Also, at l equal to zero, the value of βl will be zero so that two equations, not involving β, may be obtained. From Equations 1·44,

$$\begin{aligned} E_L &= E^+e^{j\omega t} + E^-e^{j\omega t} \\ I_L &= I^+e^{j\omega t} + I^-e^{j\omega t} \end{aligned} \tag{1·53}$$

(*Note:* Henceforth the factor $e^{j\omega t}$ will not be written in calculations—the normal procedure in the case of alternating current calculations—but will be reinserted only when necessary to the calculations.)

The values of the current waves and the voltage waves are not independent but are related—Equations 1·32—by the characteristic impedance of the line. Substituting for I^+ and I^- in the second equation of Equations 1·53, we obtain

$$\begin{aligned} E_L &= E^+ + E^- \\ I_L &= \frac{E^+}{Z_0} - \frac{E^-}{Z_0} \end{aligned} \tag{1·54}$$

However, in any impedance, the voltage divided by the value of the impedance is equal to the current or

$$Z_L = \frac{E_L}{I_L} \tag{1·55}$$

Replacing E_L and I_L by their values from Equations 1·54, we find that

$$Z_L = \frac{E^+ + E^-}{\dfrac{E^+}{Z_0} - \dfrac{E^-}{Z_0}} \tag{1·56}$$

Equation 1·56 involves only two unknowns, E^+ and E^-. E^- may now be obtained in terms of E^+, Z_0, and Z_L.

$$E^- = E^+\frac{Z_L - Z_0}{Z_L + Z_0} \tag{1·57}$$

This equation shows that the ratio of the reflected wave constant, E^-, to the incident wave constant, E^+, is dependent only on two impedances, the characteristic impedance of the transmission line and the load impedance. The symbol ρ, called the reflection factor, is used to denote the ratio shown in Equation 1·57:

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{1·58}$$

Substituting Equation 1·58 into Equation 1·57, we obtain

$$E^- = \rho E^+ \tag{1·59}$$

The ratio of I^+ to I^- is obtained by substituting I^+Z_0 for E^+ and $-I^-Z_0$ for E^-. Z_0 may be factored out so that

$$I^- = -\rho I^+ \tag{1·60}$$

The reflection factor, ρ, can be positive or negative and have any phase angle, depending on the values of Z_L and Z_0. If Z_L is a pure resistance, and in the dissipationless case Z_0 is a pure resistance, ρ is real. If Z_L is less than Z_0, ρ is negative. When ρ is negative the voltage wave is reflected with a change in sign while the current wave is reflected without a change in sign. If Z_L is larger than Z_0, ρ is positive. When ρ is positive the voltage wave is reflected without a change in sign while the current wave is reflected with a change in sign. When Z_L is complex, ρ is complex. If ρ is complex the wave will be reflected with both a change in phase equal to the phase angle of ρ and a change in magnitude, dependent on the magnitude of ρ. From Equation 1·58, we see that ρ can never have a magnitude greater than 1 as long as Z_L remains a passive impedance.

Only one remaining unknown is left to be evaulated, E^+. Its value is obtained by expressing the sending end voltage in terms of E^+. Remembering, however, that the receiving end is located at l equal to zero and any point along the line will be the negative of its distance from the receiving end, we must substitute minus l for l in the transmission line Equations 1·44. Equations 1·59 and 1·60 are also substituted into Equations 1·44:

$$\begin{aligned} V_l &= E^+e^{j\beta l} + \rho E^+e^{-j\beta l} \\ I_l &= I^+e^{j\beta l} - \rho I^+e^{-j\beta l} \end{aligned} \tag{1·61}$$

Equations 1·61 now give the voltage and current at any point l on the line, l being the absolute distance along the line from the receiving end to the point in question. To obtain both equations in terms of E^+,

E^+ over Z_0 is substituted for I^+.

$$\begin{aligned} V_l &= E^+(e^{j\beta l} + \rho e^{-j\beta l}) \\ I_l &= \frac{E^+}{Z_0}(e^{j\beta l} - \rho e^{-j\beta l}) \end{aligned} \tag{1·62}$$

The equations of Equations 1·62 show the distribution of current and voltage along the transmission line in terms of the unknown E^+. It is interesting to note that the functions (enclosed in parentheses in the equations) which determine the distribution of the current and voltage along the line at any instant are dependent only on ρ and β. These two constants are completely independent of the sending end conditions. The only value determined by the sending end, sometimes called the transmitting end, is the value of the multiplying constant E^+. To obtain the equations for the sending end voltage, E_s, and the sending end current, I_s, let l be made equal to the line length s. The electrical length of the transmission line in degrees, $\theta°$, is given by

$$\theta° = \beta s \tag{1·63}$$

where β is expressed in degrees per unit length, 2π radians being equal to 360°. $\theta°$ is called the electrical length of the transmission line, inasmuch as it is the amount of phase shift introduced into a wave as it travels down the line. Substituting Equation 1·63 into Equations 1·61, we obtain

$$\begin{aligned} E_s &= E^+(e^{j\theta°} + \rho e^{-j\theta°}) \\ I_s &= \frac{E^+}{Z_0}(e^{j\theta°} - \rho e^{-j\theta°}) \end{aligned} \tag{1·64}$$

As E_s is usually a known quantity, the equations of Equations 1·64 are simultaneous equations with two unknowns, E^+ and I_s. $e^{j\theta°}$ represents a magnitude of unity with a phase shift of $\theta°$. For convenience, $e^{j\theta°}$ is usually symbolized by $1\underline{/\theta°}$ and $e^{-j\theta°}$ by $1\underline{/-\theta°}$. Very often the degree sign is omitted and understood. Substituting these simplifications into Equations 1·64, we obtain

$$\begin{aligned} E_s &= E^+(1\underline{/\theta} + \rho\underline{/-\theta}) \\ I_s &= \frac{E^+}{Z_0}(1\underline{/\theta} - \rho\underline{/-\theta}) \end{aligned} \tag{1·65}$$

From these equations we can determine E^+ and I_s when E_s, ρ, and θ are known. ρ and θ are obtained using the line constants, the value of Z_L,

and the line length s. Solving for E^+, we find that

$$E^+ = \frac{E_s}{1\underline{/\theta} + \rho\underline{/-\theta}} \tag{1·66}$$

Once E^+ is known, the voltage and current at any point l along the line can be calculated using Equations 1·62.

The input impedance of the transmission line can be obtained directly from Equations 1·65. The input impedance is equal to the input voltage divided by the input current. Dividing E_s by I_s, we find that the E^+ values cancel out, yielding

$$Z_s = Z_0 \frac{1\underline{/\theta} + \rho\underline{/-\theta}}{1\underline{/\theta} - \rho\underline{/-\theta}} \tag{1·67}$$

Thus the input impedance of a transmission line, of length θ degrees and terminated in a load impedance Z_L, can be found by substituting into Equation 1·67.

EXAMPLE 1·4 What is the input impedance of a transmission line that has the following constants?

$$Z_0 = 150 \text{ ohms}$$
$$\beta l = 30° = \theta°$$
$$Z_L = 100 + j100 \text{ ohms}$$

The input impedance is given by Equation 1 67. However, it is first necessary to obtain the value of ρ by substituting into Equation 1·58:

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0}$$
$$\rho = \frac{100 + j100 - 150}{100 + j100 + 150}$$
$$\rho = 0.42 \underline{/94.8°}$$

This value of ρ and the value of θ, as given, can now be substituted into Equation 1·67.

$$Z_s = Z_0 \frac{1\underline{/\theta} + \rho\underline{/-\theta}}{1\underline{/\theta} - \rho\underline{/-\theta}}$$
$$Z_s = 150 \frac{1\underline{/30°} + 0.42\underline{/94\ 8°} \times 1\underline{/-30°}}{1\underline{/30°} - 0.42\underline{/94.8°} \times 1\underline{/-30°}}$$
$$Z_s = 150 \frac{1.36\underline{/39.8°}}{0.69\underline{/10°}}$$
$$Z_s = 295.5\underline{/29.8°} = 6.5\ 25 + j146.5 \qquad \textit{Ans.}$$

The answer indicates that a generator supplying a load impedance of $100 + j100$ ohms at the end of a transmission line with the above constants would actually have across its terminals an impedance of $295.5\underline{/29.8^\circ}$ ohms.

1·10 The input impedance of quarter-wave and half-wave dissipationless lines

The dissipationless transmission line, as seen from Equation 1·67, acts very much like an impedance transformer in ordinary circuit theory. With the load impedance remaining fixed, the input impedance can be varied either by varying the line constants or by varying the line length. Substituting for ρ in Equation 1·67 and multiplying numerator and denominator by $(Z_L + Z_0)$, we get

$$Z_s = Z_0 \frac{(Z_L + Z_0)\underline{/\theta^\circ} + (Z_L - Z_0)\underline{/-\theta^\circ}}{(Z_L + Z_0)\underline{/\theta^\circ} - (Z_L - Z_0)\underline{/-\theta^\circ}} \tag{1·68}$$

Two very interesting cases are obtained: one when θ° is made equal to 90°, called a quarter-wave line, the other when θ° is made equal to 180°, called a half-wave line. Substituting into Equation 1·68 for the case of the quarter-wave line, where θ is 90°, we obtain

$$Z_s = Z_0 \frac{(Z_L + Z_0)\underline{/90^\circ} + (Z_L - Z_0)\underline{/-90^\circ}}{(Z_L + Z_0)\underline{/90^\circ} - (Z_L - Z_0)\underline{/-90^\circ}} \tag{1·69}$$

However, $\underline{/90^\circ}$ is equal to j and $\underline{/-90^\circ}$ is equal to $-j$. Making these substitutions and simplifying, we find that

$$Z_s = \frac{Z_0^2}{Z_L} \tag{1·70}$$

Thus the input impedance of a dissipationless quarter-wave transmission line is equal to the characteristic impedance squared divided by the load impedance. If the load impedance is a pure resistance, the input impedance is a pure resistance. Also, since the input impedance depends upon the reciprocal of Z_L, if Z_L is capacitive, the input impedance is inductive Similarly if Z_L is inductive, the input impedance is capacitive. Using the equality that odd multiples of 90° are equal to j with the proper sign, it can be shown that Equation 1·70 is true for any odd multiple of 90°.

Where θ° is made equal to 180°, called a half-wave line, both $\underline{/\theta^\circ}$ and $\underline{/-\theta^\circ}$ are equal to -1. Substituting this value into Equation 1·68,

we obtain for the half-wave line

$$Z_s = Z_0 \frac{-(Z_L + Z_0) - (Z_L - Z_0)}{-(Z_L + Z_0) + (Z_L - Z_0)} \tag{1·71}$$

This may be simplified, yielding

$$Z_s = Z_L \tag{1·72}$$

Equation 1·72 shows that a half-wave dissipationless transmission line of any characteristic impedance simply acts to bring the load impedance terminals to a different point. Using the axiom that unity at any multiple of 180° is always equal to 1 or −1, we can show that Equation 1·72 is true for any multiple of 180°.

EXAMPLE 1·5 Design a line which will have an input impedance of 100 ohms when it has a real load impedance Z_L of 400 ohms.

In Equation 1·70 we see that a transmission line, one quarter of a wavelength long, will do the job if

$$Z_0^2 = Z_L Z_s$$

or

$$Z_0 = \sqrt{Z_L Z_s}$$

Since Z_0 in a dissipationless line is always purely resistive, a solution to this problem will be possible only when the product $Z_s Z_L$ is real and positive. Substituting the values of Z_L and Z_s from the problem, we find that

$$Z_0 = \sqrt{100 \cdot 400}$$

$$Z_0 = 200 \text{ ohms}$$

From Equation 1·48, we can use a coaxial line where

$$\log_{10} \frac{D}{d} = \frac{200}{138}$$

or

$$\log_{10} \frac{D}{d} = 1.450$$

$$\frac{D}{d} = 28.2 \qquad \textit{Ans.}$$

1·11 THE INPUT IMPEDANCE OF AN OPEN-CIRCUITED DISSIPATIONLESS LINE

In an open-circuited line, Z_L is equal to infinity. It is necessary, in Equation 1·58 for the reflection factor to divide numerator and denominator by Z_L in order to avoid the quantity infinity over infinity. The

equation for ρ then becomes

$$\rho = \frac{1 - \dfrac{Z_0}{Z_L}}{1 + \dfrac{Z_0}{Z_L}} \tag{1·73}$$

Substituting Z_L equal to infinity or Z_0/Z_L equal to zero for finite values of Z_0, we find that

$$\rho = 1 \tag{1·74}$$

Thus the reflection factor of an open-circuited line is unity. The voltage is reflected in its entirety without a change in sign while the current is reflected in its entirety with a change in sign. Substituting Equation 1·74 into Equation 1·67 for the input impedance of a transmission line, we find that

$$Z_s = Z_0 \frac{1\underline{/\theta} + 1\underline{/-\theta}}{1\underline{/\theta} - 1\underline{/-\theta}} \tag{1·75}$$

But, from elementary phasor analysis, $1\underline{/\theta}$ is equal to $(\cos\theta + j\sin\theta)$ and $1\underline{/-\theta}$ is equal to $(\cos\theta - j\sin\theta)$. Substituting these values into Equation 1·75 and simplifying, we get

$$Z_s = Z_0 \frac{2\cos\theta^\circ}{2j\sin\theta^\circ} \tag{1·76}$$

or

$$Z_s = -jZ_0 \cot\text{an}\,\theta^\circ \qquad Z_L = \infty \tag{1·77}$$

This demonstrates that the input impedance of an open-circuited dissipationless transmission line is a pure reactance and, provided it is less than a quarter wavelength long, is capacitive. A curve of Equation 1·77 is shown in Figure 1·7. We can see that the transmission line will go through successive resonant points. At all odd multiples of a quarter wavelength it will be series resonant, having an input impedance of zero, and at all even multiples of a quarter wavelength it will be parallel resonant, having an input impedance of infinity. Of course, since it is actually impossible to obtain a perfect dissipationless line, there will always be a resistive component left at the resonant points resembling the case of resonance in lumped constant theory.

It is often useful to employ line segments as reactive elements in circuits. As shown by Equation 1·77,

$$Z_s = -jX_C \qquad Z_L = \infty, \quad \theta < \frac{\lambda}{4} \tag{1·78}$$

where

$$X_C = Z_0 \operatorname{cotan} \theta^\circ \tag{1·79}$$

Substituting the expression for

$$X_C = \frac{1}{2\pi f C} \tag{1·80}$$

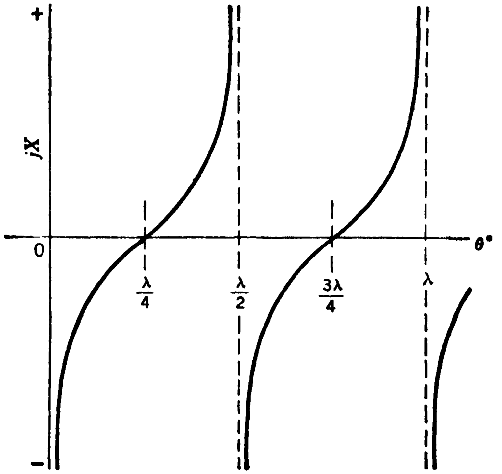


FIG. 1·7 The input impedance of an open-circuited dissipationless transmission line.

into Equation 1·79, we find that

$$\frac{1}{2\pi f C} = Z_0 \operatorname{cotan} \theta^\circ \tag{1·81}$$

or

$$C = \frac{1}{2\pi f Z_0 \operatorname{cotan} \theta^\circ} \tag{1·82}$$

Here f is the frequency and θ° the electrical length. Thus it is possible, at ultrahigh frequencies, to obtain the equivalent of an accurate value of capacitance with an air line. This capacity differs from a lumped capacity; it is not independent of f. As θ° varies between zero and $\lambda/4$, it will result in a capacity which will vary between C equal to zero and C equal to infinity. For longer lengths, the reactance is capacitive when cotan θ° is positive and inductive when cotan θ° is negative. To better appreciate what is happening on an open-circuited line, it is ad-

visable to refer to the original wave equations. Using these, we now obtain the phasor diagrams of Figure 1·8. Both E^+ and I^+ are at the phase angle $\theta°$, and

$$I^+ = \frac{E^+}{Z_0} \tag{1·83}$$

The wave has to travel down the line to the end and then the reflected wave travels back to the input. It means that at any instant

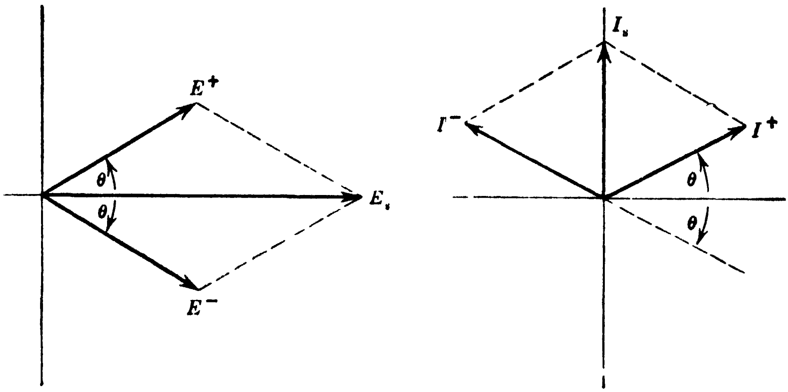


FIG. 1·8 The phasor diagrams for the waves at the input to an open-circuited transmission line

the reflected wave at the input will be lagging the propagating wave by $2\theta°$. Also, from the boundary conditions, the current at the end of an open line must be zero. Thus I^- must be the negative of I^+. Since

$$I^+ = -\rho I^- \tag{1·84}$$

ρ must equal 1. Thus at the open end, I^- has an added phase shift of 180° and E^- is equal to E^+. Thus I^- in the phasor* diagram is shown lagging I^+ by 180° less $2\theta°$ while E^- is shown lagging by E^+ by $2\theta°$. Repeating this process along the line at the points $(\theta - \beta l)$, where l is zero at the sending end, we find that E and I vary in amplitude as shown in Figure 1·9; the phase of E is either 0 or 180° and the phase of I is either 90° or 270°. The variation in amplitude with distance is known as a standing wave.

* Phasor diagrams are referred to as vector diagrams in many textbooks. To avoid an ambiguous use of the term "vector," since true vectors are an important consideration in ultrahigh frequency transmission and radiation, the device for representing a sine wave by an arrow is referred to as a phasor.

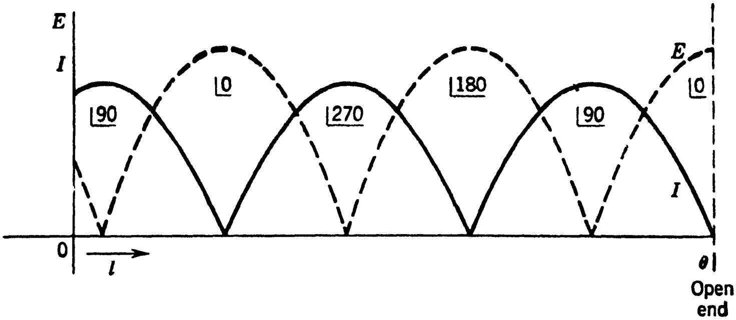


FIG. 1·9 Distribution of current and voltage along an open-circuited line.

EXAMPLE 1·6 Given a transmission line with a characteristic impedance Z_0 equal to 100 ohms, what electrical length is necessary to obtain a C of $10\mu\mu$ farads at 100 megacycles? Neglect dissipation.

From Equation 1·82, using an open-circuited line, we obtain

$$\cotan \theta = \frac{1}{2\pi f Z_0 C}$$

or

$$\cotan \theta = \frac{1}{6.28(100 \times 10^6)(100)(10 \times 10^{-12})}$$

$$\cotan \theta = \frac{1}{0.628}$$

$$\cotan \theta = 1.59$$

$$\theta = 32.08° \qquad \textit{Ans.}$$

1·12 THE INPUT IMPEDANCE OF A SHORT-CIRCUITED DISSIPATIONLESS LINE

For a short-circuited load, Z_L is equal to zero. Substituting into Equation 1·58 for ρ, we obtain

$$\rho = -1 \tag{1·85}$$

Now substituting Equation 1·85 into Equation 1·67, the equation for the input impedance of a transmission line, we find that

$$Z_s = Z_0 \frac{1\underline{/\theta°} - 1\underline{/-\theta°}}{1\underline{/\theta°} + 1\underline{/-\theta°}} \tag{1·86}$$

By substituting for $1\angle\theta$ and $1\angle-\theta$ as before, we obtain

$$Z_s = Z_0 \frac{\cos\theta^\circ + j\sin\theta^\circ - (\cos\theta^\circ - j\sin\theta^\circ)}{\cos\theta^\circ + j\sin\theta^\circ + \cos\theta^\circ - j\sin\theta^\circ} \tag{1·87}$$

so that

$$Z_s = j\,Z_0 \tan\theta^\circ \tag{1·88}$$

This shows that the input impedance of a short-circuited dissipationless line is a pure reactance and, if it is less than a quarter of a wave-

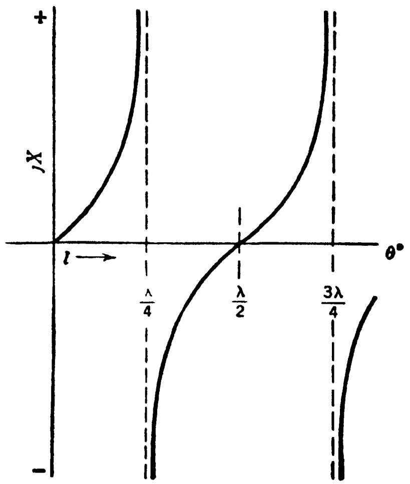


FIG. 1·10 Graph of the input impedance of a short-circuited line as its length l is varied.

length long, is inductive. A curve of Equation 1·88 is shown in Figure 1·10. This line will also go through successive resonant points. It is just the opposite of the results for the open-circuited line. At all lengths which are odd multiples of a quarter wavelength, the input impedance will go through a parallel resonance of infinite impedance and at all lengths which are even multiples it will go through a series resonance of zero impedance. Since the impedance is inductive,

$$Z_s = jX_L \qquad Z_L = 0, \qquad \theta < \frac{\lambda}{4} \tag{1·89}$$

where

$$X_L = Z_0 \tan\theta^\circ \tag{1·90}$$

But X_L is usually expressed by

$$X_L = 2\pi f L \tag{1·91}$$

where L is the inductance in henrys at the frequency f. This inductance differs from the lumped constant concept inasmuch as it varies with frequency. Returning to Equation 1·91, we obtain

$$2\pi fL = Z_0 \tan \theta^\circ \tag{1·92}$$

so that

$$L = \frac{Z_0 \tan \theta^\circ}{2\pi f} \tag{1·93}$$

(Equation 1·93 is applicable only when $\tan \theta^\circ$ is positive. When $\tan \theta^\circ$ is negative, the impedance is capacitive.) Thus, at ultrahigh frequencies, it is possible to use lengths of shorted transmission lines for inductive elements.

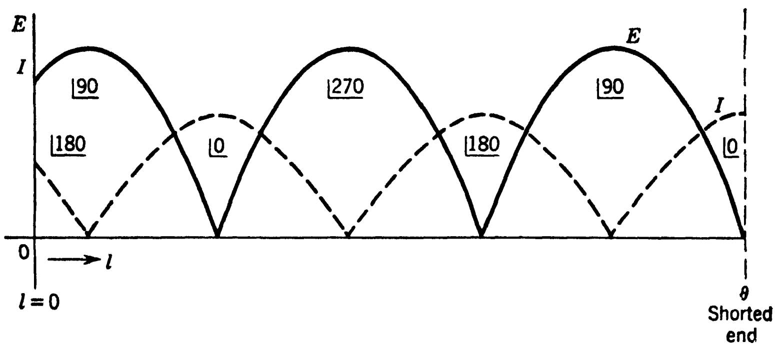


FIG. 1·11 The distribution of current and voltage along a transmission line shorted at its receiving end.

By applying the same reasoning to the propagated and reflected waves that was used for the open-circuited line, it is possible to obtain the voltage and current distribution along a shorted line. The voltage at the shorted end must be zero and the current a maximum. A curve of the current and voltage distribution is shown in Figure 1·11. In this case the current and voltage are also 90° out of phase at all points. It should be noted that the ratio of maximum to minimum is infinite.

EXAMPLE 1·7 Given a transmission line with a characteristic impedance of 100 ohms, what electrical length is necessary to obtain an inductance of 0.2 microhenry at 100 megacycles?

From Equation 1·93, for a shorted line,

$$\tan \theta = \frac{2\pi fL}{Z_0}$$

and

$$\tan\theta = \frac{6.28(100 \times 10^6)(0.2 \times 10^{-6})}{100}$$

$$\tan\theta = 1.256$$

or

$$\theta = 51.5° \qquad \textit{Ans.}$$

1·13 Standing waves on transmission lines

The variation of amplitude along the length of a transmission line, caused by the difference in phase between the propagated and reflected wave, is known as a standing wave. Equation 1·61 gives the voltage at

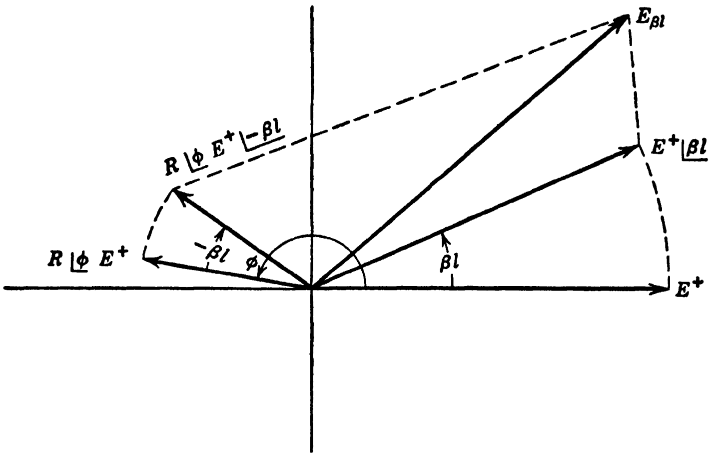


Fig. 1·12 Phasor diagram of standing waves on transmission lines showing the propagated and reflected voltage waves and their resultant sum.

any point l units from the end of a transmission line. If, instead, the same equation is rewritten referring to any point βl degrees from the the receiving end, the equation becomes

$$E_{\beta l} = E^+\underline{/\beta l} + \rho E^+\underline{/-\beta l} \qquad (1\cdot94)$$

We can see from Equation 1·94 that one of the components of the voltage is made up of a propagated wave that has to travel an electrical distance βl degrees to the receiving end. This is shown in Figure 1·12. At the load end, the wave is modified by the factor ρ, which has both magnitude and phase. The factor ρ may be written

$$\rho = R\underline{/\phi} \qquad (1\cdot95)$$

where R is the magnitude factor and ϕ the phase factor.

Equation 1·94 is illustrated in the phasor diagram of Figure 1·12. After the wave is modified by ρ at the receiving end, the reflected wave has to travel back an electrical distance βl degrees as shown in the figure. The two waves are then added at the point l. We can see from the diagram that when βl is equal to one half of ϕ, the reflected wave and propagated wave will be in phase. It results in a maximum voltage equal to $(1 + R)E^+$. At the point where βl is equal to $(\frac{1}{2}\phi + 90°)$, RE^+ will be 180° out of phase with E^+. At this point the voltage will be equal to $(1 - R)E^+$ This latter variation in voltage is shown in Figure 1·13.

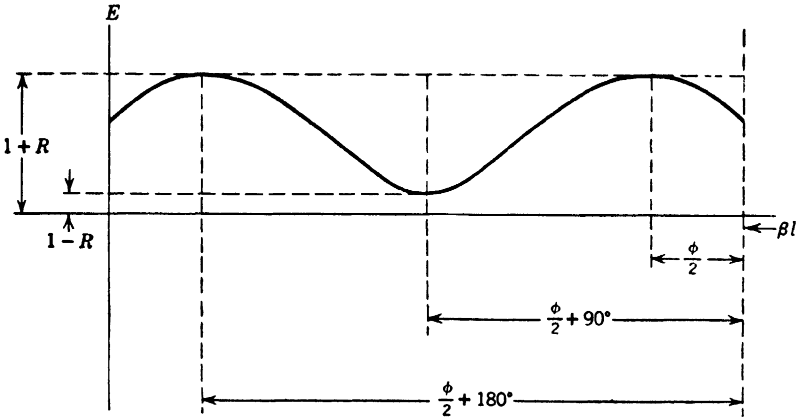


FIG. 1·13 A voltage standing wave on a transmission line showing how the positions of the maxima and minima are related ϕ.

A variation in amplitude, a standing wave, is absent only on those transmission lines where ρ is equal to zero. Equating ρ to zero in the equation for ρ, we find that Equation 1·58 becomes

$$0 = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{1.96}$$

or

$$Z_L = Z_0 \tag{1·97}$$

When no standing waves are present on the transmission line, it is said to be matched. Referring to Equation 1·97, we find that the transmission line is matched only when the load impedance is equal to the characteristic impedance. In the dissipationless case the characteristic impedance is a pure resistance showing that the termination also has to be a pure resistance to match the line. When the trans-

mission line is matched, since only the propagated wave is present, the voltage along the line is constant in amplitude but varies in phase, the phase angle being equal to βl degrees. When the line is terminated in any pure resistance, and since Z_0 is a pure resistance, ϕ will be either 0° or 180°. This follows because

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} = \pm R \quad (\text{when } Z_L = \text{pure resistance}) \tag{1·98}$$

Thus ϕ will be 0° when Z_L is greater than Z_0 and ϕ will be 180° if Z_L is less than Z_0. Referring now to Figure 1·13, we find that when ϕ is zero (Z_L is a pure resistance greater than Z_0) $\phi/2$ is zero and a minimum occurs 90 electrical degrees from the end of the line. If ϕ is equal to 180° (Z_L is a pure resistance less than Z_0), $\phi/2$ is equal to 90° so that a maximum occurs 90° from the end of the line. Calling the ratio of E_{max} to E_{min}, S, when ϕ is zero, we find that

$$S = \frac{E_{max}}{E_{min}} = \frac{1 + R}{1 - R} \tag{1·99}$$

From equation 1·98 for the case where Z_L is a pure resistance,

$$R = \frac{\dfrac{Z_L}{Z_0} - 1}{\dfrac{Z_L}{Z_0} + 1} \tag{1·100}$$

Substituting Equation 1·100 into Equation 1·99, we obtain

$$S = \frac{1 + \dfrac{\dfrac{Z_L}{Z_0} - 1}{\dfrac{Z_L}{Z_0} + 1}}{1 - \dfrac{\dfrac{Z_L}{Z_0} - 1}{\dfrac{Z_L}{Z_0} + 1}} = \frac{Z_L}{Z_0} \tag{1·101}$$

so that if a minimum occurs 90° from the load,

$$Z_L = SZ_0, \quad Z_L = \text{pure resistance} \tag{1·102}$$

Similarly, if the maximum occurs 90° from the load, R is negative and

$$Z_L = \frac{Z_0}{S}, \quad Z_L = \text{pure resistance} \tag{1·103}$$

When Z_L is not a pure resistance, the phase angle, ϕ, of ρ is given by twice the electrical distance from the load to E maximum, and R of ρ, from Equation 1 99, is given by

$$R = \frac{S - 1}{S + 1} \tag{1 104}$$

Substituting these results into Equation 1 98, we find that

$$R\underline{/\phi} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{1 105}$$

or

$$Z_L = Z_0 \frac{1 + R\underline{/\phi}}{1 - R\underline{/\phi}} \tag{1 106}$$

Using Equation 1 106, we can determine the value of an impedance by terminating a transmission line in the impedance, measuring the standing wave ratio on the transmission line, and noting the position of the minimum The minimum is used (as shown in Figure 1 13) because its location is much more definite and the minimum is easier to detect than the maximum

1·14 Transmission Lines Used as Measuring Devices

A schematic circuit of the apparatus necessary to make standing wave measurements for the determination of impedances is shown in

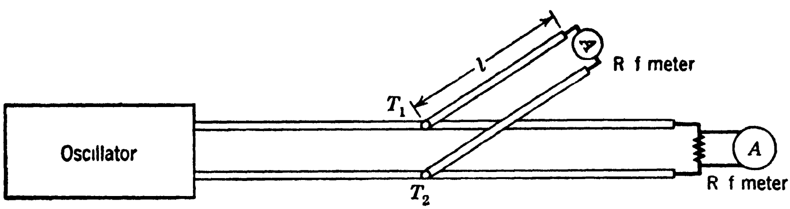


Fig 1 14 Voltmeter measuring apparatus for detecting standing waves on a transmission line

Figure 1 14 It consists of an oscillator, which is a source of power, a standard transmission line, and a means for detecting the standing wave on the transmission line, usually called a probe. The probe is actually an r-f voltmeter with a high input impedance

The transmission line has to be very carefully constructed so that the correct characteristic impedance is obtained The characteristic impedance must be kept as constant as possible along the length of the

line; consequently, the tolerances during construction must be kept very close. The construction of a coaxial measuring line and of a balanced measuring line are shown in Figure 1·15. We see that both types are completely shielded except for a slot through which the probe is inserted. The length of line available for measurement should be greater than a half wavelength at the lowest frequency used. Thus

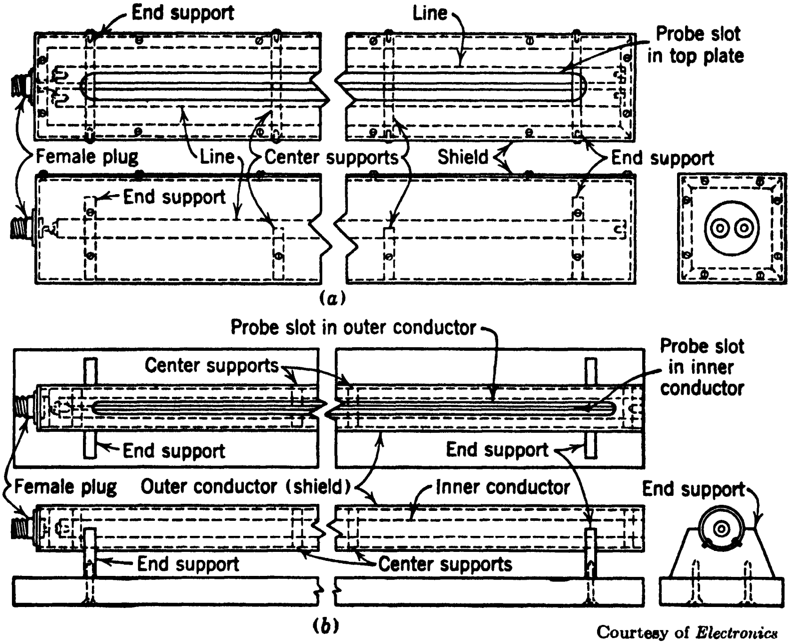


FIG. 1·15 (*a*) A balanced line used for impedance measurements, constructed in a trough for shielding. (*b*) A coaxial line with a slot in the outside shield used for impedance measurements.

at least one maximum and one minimum will always be available on the line for measurement.

The probe construction will vary with different frequency bands. At the low radio frequencies, the ordinary commercial vacuum tube voltmeter usually has a high enough impedance to allow it to be used directly across the transmission line. A very simple type of ultrahigh frequency voltmeter is illustrated in Figure 1·14. An ordinary transmission line of length l is employed with a very low impedance r-f ammeter connected across one end. When l is equal to $\frac{1}{4}\lambda$ at the frequency being used, the impedance looking into its input end, across T_1T_2, is very high. These points of contact with the line are moved

along the line, and the voltage at points along the line, the standing wave pattern, will be obtained. For a more sensitive device the end where the r-f meter is located may be shorted and a loosely coupled output loop used to conduct a small amount of r-f energy to a receiver used as the indicator. Figure 1·16 shows the construction of the dual and coaxial types of probe.

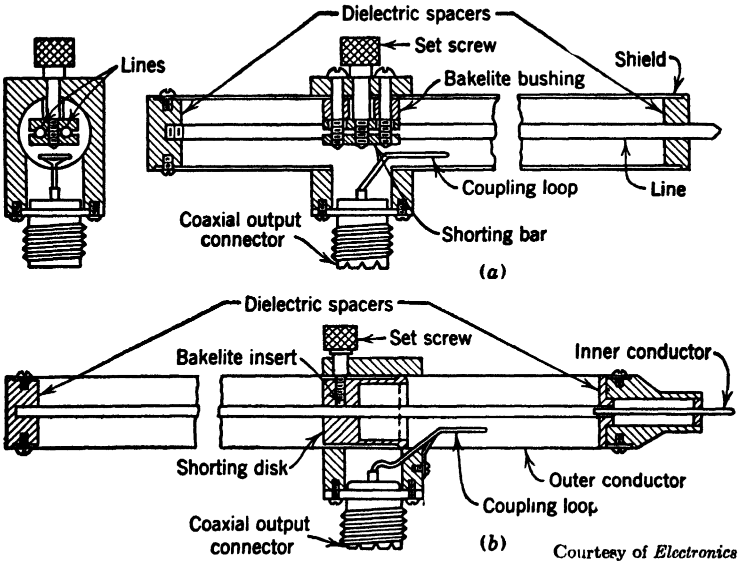
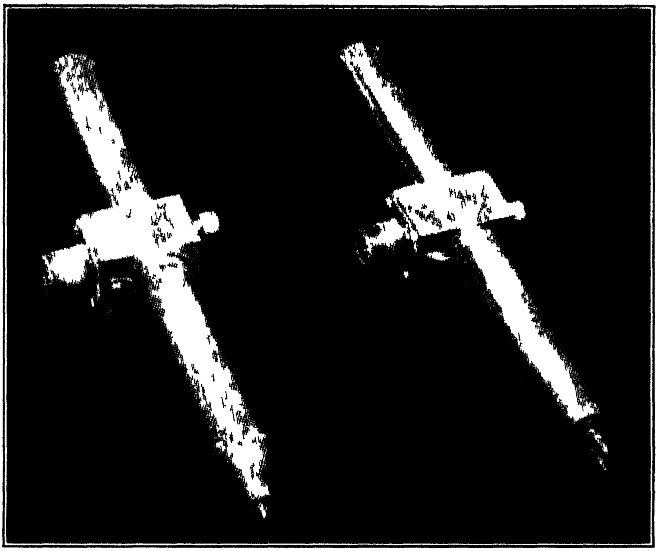


FIG. 1·16 (*a*) A dual type probe for standing wave measurements on a balanced transmission line. (*b*) A coaxial probe with a shorting disk and pickup loop inside the outer conductor of the probe.

The dual probe is always used by making contact with the standard transmission line at the probe's input points. The coaxial probe may be used by having its inner conductor come in contact with the inner conductor of the standard transmission line. The shields of both are always in electrical contact with the shields of the lines. Very often, however, the inner conductor of the coaxial line probe is merely projected a small distance into the shield of the coaxial standard line, acting like a small vertical open-ended antenna. When this is done the position and length of the probe's inner conductor inside the measuring line must be maintained constant as the probe is moved along the line. This necessitates the use of some type of sliding carriage for the probe.

For convenience, the shorts on the probes are usually made variable. This may be accomplished by means of a thumb screw which operates

through a slot in the side of the probe shield. The physical appearances of two probes of this type are shown in Figure 1·17



Courtesy of *Electronics*

FIG 1 17 A photograph of the two probes shown diagrammatically in Figure 1·16.

When a standard transmission line and its associated probe are used to measure an impedance at ultrahigh frequencies, there are four steps to be followed:

1. A transmission line of known characteristic impedance is first set up.
2. The end of the transmission line is then shorted and, with a high impedance probe, the position of the minimum is determined. This minimum will be 180° from the end.
3. The unknown impedance is connected at the end and the position of the minimum is determined. If the minimum shifts toward the load, $\phi/2$ will be equal to 90° minus the shift. If the minimum shifts toward the generator, $\phi/2$ will be equal to 90° plus the shift.
4. The standing wave ratio, S, is measured on the line and from Equation 1·104, R, the magnitude of ρ, is determined.

EXAMPLE 1·8 What is the value of the load impedance terminating a transmission line, whose characteristic impedance is equal to 100 ohms, when the standing wave ratio on the line is 3? The position of the minimum

is determined with the line shorted, and when the load is connected the minimum shifts 30° towards the generator.

From step 3 of the preceding paragraph, $\phi/2$ is equal to 120°, or ϕ is equal to 240°.

From Equation 1·104,

$$R = \frac{3 - 1}{3 + 1} = 0.5$$

Substituting into Equation 1·106, we find that

$$Z_L = 100 \frac{1 + 0.5\underline{/240^\circ}}{1 - 0.5\underline{/240^\circ}}$$

$$Z_L = 100 \frac{0.866\underline{/-30^\circ}}{1.325\underline{/19.1^\circ}}$$

$$Z_L = 65.2\underline{/-49.1} \text{ ohms} \qquad \textit{Ans.}$$

1·15 Single-stub transmission line matching

To keep standing waves from forming on a transmission line it is necessary to match the line. This can be done by any means which transforms the load impedance Z_L to the characteristic impedance Z_0.

A simple method of matching a line employs a single stub on the transmission line. The stub consists of an open or shorted section of a transmission line. It is attached to the line at the point where the impedance looking into the line, in parallel with the stub, has a real component of admittance equal to $1/Z_0$. $\theta°$ from the load Z_L, the input impedance, Z_s, of a transmission line is—from Equation 1·67—

$$Z_s = Z_0 \frac{\cos\theta + j\sin\theta + \rho\cos\theta - j\rho\sin\theta}{\cos\theta + j\sin\theta - \rho\cos\theta + j\rho\sin\theta} \tag{1·107}$$

This gives for the admittance, Y_s,

$$Y_s = \frac{1}{Z_0}\frac{(1 - \rho)\cos\theta + j(1 + \rho)\sin\theta}{(1 + \rho)\cos\theta + j(1 - \rho)\sin\theta} \tag{1·108}$$

Removing the imaginary component from the denominator by multiplying numerator and denominator by the conjugate of the denominator, we obtain

$$Y_s = \frac{1}{Z_0}$$
$$\frac{(1+\rho)(1-\rho)\cos^2\theta + (1-\rho)(1+\rho)\sin^2\theta + j\cos\theta\sin\theta\,[(1+\rho)^2 - (1-\rho)^2]}{(1 + \rho)^2\cos^2\theta + (1 - \rho)^2\sin^2\theta} \tag{1·109}$$

or

$$Y_s = \frac{1}{Z_0} \frac{(1 - \rho^2)(\cos^2\theta + \sin^2\theta) + j\,(\cos\theta\sin\theta)4\rho}{(1 + \rho^2)(\cos^2\theta + \sin^2\theta) + 2\rho\,(\cos^2\theta - \sin^2\theta)} \quad (1\cdot110)$$

But

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \cos^2\theta - \sin^2\theta &= \cos 2\theta \end{aligned} \quad (1\cdot111)$$

Equation 1·111 may be substituted into Equation 1·110 to obtain

$$Y_s = \frac{1}{Z_0}\left[\frac{1 - \rho^2}{1 + \rho^2 + 2\rho\cos 2\theta} + j\frac{4\rho\cos\theta\sin\theta}{1 + \rho^2 + 2\rho\cos 2\theta}\right] \quad (1\cdot112)$$

In order for the impedance looking into the line to be Z_0, the real part of Equation 1·112 should be equal to $1/Z_0$ and the imaginary part should be canceled out by the admittance of the stub. Equating the real part of Equation 1·112 to $1/Z_0$, we obtain

$$\frac{1 - \rho^2}{1 + \rho^2 + 2\rho\cos 2\theta} = 1 \quad (1\cdot113)$$

or

$$\theta^\circ = \frac{\cos^{-1}(-\rho)}{2} \quad (1\cdot114)$$

Since Equation 1·114 is a solution when ρ is real, θ° may be measured from a minimum of the voltage standing wave. At this minimum point V and I will be in phase and the effective ρ at this point, designated as ρ_M, will be real. In terms of the standing wave ratio, where S is greater than one, ρ_M is obtained by substituting Equation 1·103 into Equation 1·98:

$$\rho_M = \frac{1 - S}{1 + S} \quad (1\cdot115)$$

Substituting Equation 1·115 into Equation 1·114 and clearing the minus sign from in front of ρ_M, we find that

$$\theta = \frac{\cos^{-1}\left(\dfrac{S - 1}{S + 1}\right)}{2} \quad (1\cdot116)$$

Equation 1·116 now gives, in terms of the standing wave ratio, the distance in electrical degrees from the minimum of the voltage standing wave ratio to the point at which the stub should be put on. Equating the imaginary part of Equation 1·112 to $1/jX_s$ will give the react-

ance at that point which has to be tuned out in order to result in a pure resistance equal to Z_0:

$$\frac{1}{jX_s} = \frac{j}{Z_0} \frac{4\rho \cos\theta \sin\theta}{1 + \rho^2 + 2\rho \cos 2\theta} \tag{1·117}$$

The reactance values of a stub are given by

$$\begin{aligned} jX_L &= jZ_0 \tan \theta_{ss}, \qquad \text{shorted stub} \\ -jX_C &= -jZ_0 \operatorname{cotan} \theta_{so}, \qquad \text{open stub} \end{aligned} \tag{1·118}$$

where the Z_0 of the stub is equal to the Z_0 of the line being "stubbed" or "flattened."

Since Equation 1·116 will have two solutions less than 90°, one positive and one negative, Equation 1·117 will also have two solutions. When $\theta°$ of Equation 1·116 is taken positive, Equation 1·117 will yield a positive X_s. Thus an open stub would be required to produce the necessary negative reactance and $\theta°$ would be measured from the minimum towards the generator. When $\theta°$ of Equation 1·116 is taken negative, Equation 1·117 will yield a negative X_s. Thus a shorted stub would be required to produce the necessary positive reactance and $\theta°$ would be measured from the minimum towards the load.

For the open stub, from Equations 1·118 and 1·117

$$\theta_{so} = \operatorname{cotan}^{-1} \frac{1 + \rho^2 + 2\rho \cos 2\theta}{4\rho \cos\theta \sin\theta} \tag{1·119}$$

Substituting the following equations—from Equation 1·114 and trigonometric identities—

$$\begin{aligned} \cos 2\theta &= -\rho \\ \cos\theta &= +\sqrt{\frac{1-\rho}{2}} \\ \sin\theta &= +\sqrt{\frac{1+\rho}{2}} \end{aligned} \tag{1·120}$$

into Equation 1·119, we obtain

$$\theta_{so}° = \operatorname{cotan}^{-1} \frac{\sqrt{1-\rho^2}}{2\rho} \quad (\theta \text{ measured from minimum towards generator}) \tag{1·121}$$

For a shorted stub, when $\theta°$ is taken negative

$$\theta_{ss} = \tan^{-1}\left(-\frac{1 + \rho^2 + 2\rho \cos 2\theta}{4\rho \cos\theta \sin\theta}\right) \tag{1·122}$$

Substituting

$$\cos 2\theta = -\rho$$

$$\cos \theta = +\sqrt{\frac{1-\rho}{2}} \tag{1·123}$$

$$\sin \theta = -\sqrt{\frac{1+\rho}{2}}$$

into Equation 1·122, we obtain

$$\theta_{ss} = \tan^{-1} \frac{\sqrt{1-\rho^2}}{2\rho} \quad (\theta \text{ measured from the minimum towards the load}) \tag{1·124}$$

EXAMPLE 1·9 A load connected to the receiving end of a transmission line produces a standing wave ratio of 2. What stub should be put on the line to eliminate the reflected wave?

From Equation 1·116,

$$\theta^\circ = \frac{\cos^{-1}\left[\dfrac{2-1}{2+1}\right]}{2}$$

$$\theta^\circ = \frac{\cos^{-1} 0.33}{2}$$

$$\theta^\circ = \frac{70.5^\circ}{2}$$

$$\theta^\circ = 35.25^\circ \qquad \textit{Ans. (a)}$$

The stub should be 35 25 electrical degrees from the minimum. For an open stub, θ° should be measured from the voltage minimum towards the generator. From Equation 1·121, since ρ is equal to ⅓,

$$\theta_{so} = \cot\text{an}^{-1} \frac{\sqrt{1-\frac{1}{9}}}{(2)(\frac{1}{3})}$$

$$\theta_{so} = \cot\text{an}^{-1} 1.425$$

$$\theta_{so} = 35^\circ \qquad \textit{Ans. (b)} \ (\theta \text{ positive})$$

For a shorted stub, θ° should be measured from the voltage minimum toward the load. From Equation 1·124,

$$\theta_{ss}^\circ = \tan^{-1} 1.425$$

$$\theta_{ss}^\circ = 55^\circ \qquad \textit{Ans. (c)} \ (\theta \text{ negative})$$

1·16 Procedure for putting a single matching stub on the line

In case it is necessary to design matching stubs which result in a standing wave ratio of less than about 1.2, the equations will not adequately solve the problem, since "end effects" and losses are not taken into consideration. The equations, and charts if they are available, are the first approximations, and a definite procedure should be followed to obtain the final desired values. A recommended procedure is the following:

1. A good high impedance voltmeter should be used to measure the standing wave ratio.
2. From the available equations or charts, the length of stub necessary should be obtained and connected at the point indicated.
3. The stub will usually only roughly "flatten the line." The length of the stub should be varied until a maximum or minimum occurs 90° away from the stub towards the generator; thus the reactive component has been canceled out. If a maximum occurs 90° away from the stub towards the generator, it means that the impedance is too low. A shorted stub should be moved slightly toward the load or an open stub should be moved slightly toward the generator. If a minimum occurs 90° away from the stub towards the generator, it means that the impedance is too high so that a shorted stub should be moved slightly towards the generator or an open stub should be moved slightly towards the load.
4. Steps 2 and 3 should be repeated until the required flatness is obtained.

1·17 Double stub transmission line matching

Single stub matching entails varying the point of connection of the stub to the line. It is sometimes inconvenient especially in coaxial transmission lines where an opening in the shield is necessary to admit the stub. The shield of the line and the shield of the stub then have to be joined in a good electrical connection to complete the circuit. By using two stubs of variable length, the need for moving the points of connection of the stubs to the line is eliminated.

Figure 1·18 shows a schematic diagram of a double stub transmission line matching circuit. The transmission line of characteristic impedance, Z_0, is terminated in an impedance Z_L. Stub 1, of length θ_1, is located at point AB on the line. Stub 2, of length θ_2, is located at point CD on the line. They are separated by a distance ϕ. This distance cannot be a half wavelength since that would effectively con-

nect the two stubs in parallel. For the same reason the stubs should not be too close together.

Let us consider now what happens on the line. At the point AB the input impedance of the line, looking toward the load, has a definite value, Z_a. Stub 1 is a reactance in parallel with this input impedance. As the length of this stub is varied, the standing wave on the rest of the line varies. It is finally adjusted so that the real part of the admittance, Y_c, at the point CD is equal to $1/Z_0$. The admittance Y_c is the admittance at the point CD looking into the transmission line towards stub 1. Stub 2 is then adjusted in length until the reactive component of the admittance is tuned out, resulting in the input admittance becoming real and equal to $1/Z_0$. Thus the input impedance is Z_0 and the line is matched.

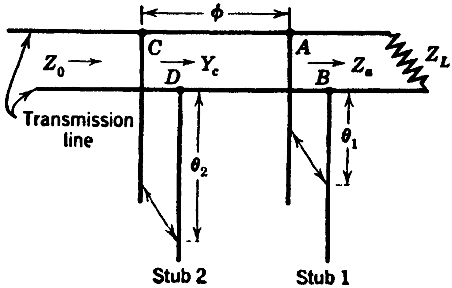


FIG. 1·18 Schematic diagram of the double stub method of matching a transmission line.

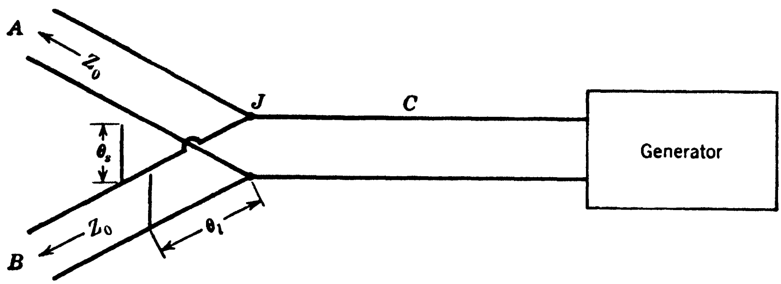


FIG. 1·19. The division of power between transmission lines by using a stub to divide the power.

1·18 Division of power between transmission lines

It is often necessary tc divide the power being transmitted by a transmission line between two other transmission lines. This division of power usually has a definite ratio, determined by the loads that are to be fed. In Figure 1·19 are shown two transmission lines, A and B. A has to obtain K times the power delivered to B. They are both obtaining their power from transmission line C. All three transmission lines are assumed to have the same characteristic impedance Z_0. Also both

transmission line loads have been matched so that there is no reflected wave; hence the impedance looking towards the load at any point on A or B is Z_0.

The technique of dividing the power is to attach a stub somewhere on one of the lines so that the line presents at its input, at the junction J, an impedance which is real and is K times Z_0. Thus, at the junction, there will be a network consisting of Z_0 and KZ_0 in parallel, and the power will divide accordingly. To produce an impedance of KZ_0 looking into line B at J, a stub should be connected to line B introducing a standing wave ratio of K. Since the stubs given by Equations 1·119 and 1·121 are used to remove standing wave ratios, then, by reciprocity, if there is no standing wave on the line these stubs, when attached at the correct point, will create a standing wave ratio of S, S being obtained from Equation 1·115. From Equations 1·102 and 1·103, at the minimum voltage point the Z_{in} of the line is given by

$$Z_{in} = \frac{Z_0}{S} \tag{1·125}$$

At the maximum voltage point the Z_{in} of the line is given by

$$Z_{in} = SZ_0 \tag{1·126}$$

To obtain a power ratio of K, S is made equal to K. Line B is joined to line A, where the standing wave on B is at a maximum. Line B will have an input impedance K times as large as the input impedance of line A, and line A will receive K times the power of line B.

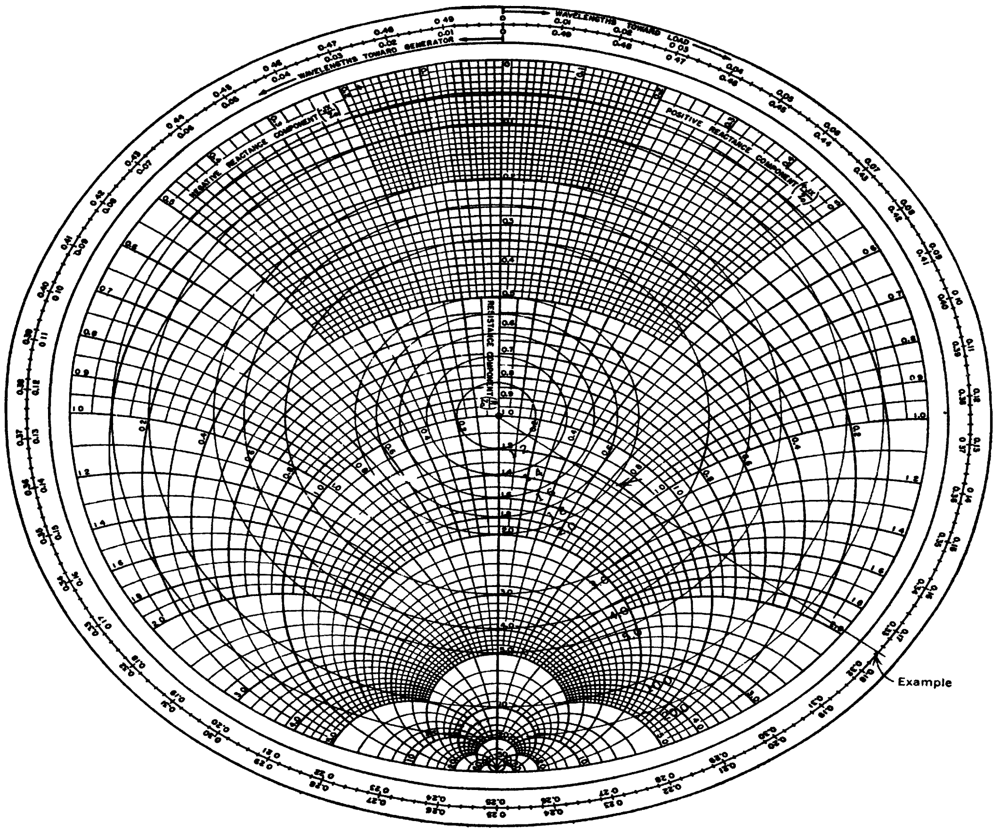
The generator line will now have a standing wave on it. A stub should be used on the generator line if no standing wave may be tolerated at the input.

The phase relations of the two lines, A and B, may now be adjusted by adjusting the lengths of the "flat" portions of the lines. The "flat" portion of the line is used because the phase shift along a "flat" line is always proportional to its length whereas the phase shift along a line with a standing wave on it is not inasmuch as the impedance varies along its length.

1·19 Circle diagram calculator

Calculations for ultrahigh frequency lines are often simplified by the use of graphs or families of curves. The curves that result, if they are families of circles, are usually referred to as circle diagrams. These diagrams are of great help particularly when the calculations have to be repeated a great many times.

One type of circle diagram is shown in Figure 1·20. It is actually



(This chart is based on the P H Smith chart from the January, 1939, *Electronics*)

FIG 1 20 Circle diagram for use in the measurements of impedances with slotted transmission lines (See section 1 19)

a special impedance coordinate system so arranged as to portray the relationships of impedances along an open wire or coaxial transmission line. The reactive and resistive values are plotted as fractions of the characteristic impedance Z_0 of the transmission line used in the measurements. Thus the results obtained on the curve are per unit values and have to be multiplied by the value of the characteristic impedance to obtain the result in ohms

The first set of lines on the calculator consists of a series of circles with their diameters coinciding along the vertical axis. These circles are the resistance component circles, and their values are given at the intersection of each circle with the center vertical axis. As shown, all are tangent at the bottom of the diagram. The second set of lines consists of the arcs of circles which originate at the very bottom of the diagram and terminate at the outer rim. They are the reactance component curves and are calibrated at the outer rim. The lines to the left of center are negative reactance lines and those to the right of center are positive reactance lines. The third set of lines, concentric circles shown in red to facilitate their use, are the standing wave ratio curves. They are calibrated from 1.2 to 20, being tangent to the top of the resistance component circles of the same value. Around the edge of the diagram are two calibrations. They are fractions of a wavelength toward the generator and toward the load.

The circle diagram can be used to obtain the values of unknown impedances in slotted line impedance measurements. A slotted line of known characteristic impedance is used with a means of obtaining the standing wave measurements. The following procedure is used:

1. The end of the slotted line is shorted and the voltage minimums are located and marked.
2. The unknown impedance is connected across the end of the slotted line and the standing wave ratio obtained. This standing wave ratio will determine the red circle on which the answer will be found.
3. The distance that the minimum of the standing wave has shifted from its position in step 1 is measured. Any minimum noted in step 1 can be used. If the minimum is measured as shifting towards the load, its measurement, as a fraction of the wavelength being used, will be noted on the scale labeled "Wavelengths toward. Load." If the minimum is measured as shifting toward the generator, its measurement will be noted on the scale labeled "Wavelengths toward Generator." A straight line can then be drawn from the center of the diagram to the fraction noted on the proper scale.

4. The intersection of the circle obtained in step 2 and the straight line obtained in step 3 marks the answer. The values of the resistance and reactance component lines which intersect at that point, multiplied by the characteristic impedance of the measuring line, yield the components of the unknown impedance being measured.

EXAMPLE 1·10 A slotted transmission line with a characteristic impedance of 50 ohms is used to measure the value of an unknown impedance. The transmission line is shorted and all the minimum positions noted. The impedance is then placed across the end of the line and a standing wave is obtained which has a maximum to minimum ratio of 2. The distance the minimum shifts is measured as shifting 0.175 wavelength towards the load. What is the value of the unknown impedance?

Referring now to Figure 1·20, we locate the answer on the red circle labeled 2.0. A straight line is drawn from the center to 0.175 on the wavelengths toward load scale. This straight line and the circle labeled 2.0 intersect at R/Z_0 equal to 1.22 and at jX/Z_0 equal to 0.725. Since the characteristic impedance of the measuring line is 50 ohms, these two values are multiplied by 50, giving

$$Z = 61.0 + j36.25 \text{ ohms} \qquad \textit{Ans.}$$

1·20 LINES WITH DISSIPATION

Only in electrically long lines or special applications of lines does dissipation have to be taken into account at ultrahigh frequencies. In low-loss lines with dissipation, a phase shift along the line plus an attenuation factor will be found. From Equation 1·31, letting βl be equal to θ—the electrical length of the line—we get

$$\begin{aligned} V &= E^+e^{-\alpha l}\underline{/-\theta} + E^-e^{\alpha l}\underline{/\theta} \\ I &= I^+e^{-\alpha l}\underline{/-\theta} + I^-e^{\alpha l}\underline{/\theta} \end{aligned} \tag{1·127}$$

Again, for convenience of calculation, assume that θ is measured from the load end of the line. The equations for the voltage at the end of the line, where θ is equal to zero, are the same as in Equations 1·54 so that

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{1·128}$$

Thus the value of ρ is the same as in the dissipationless line. Substituting Equation 1·128 in Equation 1·127 and with l measured from the receiving end of the line in a negative direction, we obtain

$$\begin{aligned} V &= E^+(e^{\alpha l}\underline{/\theta} + \rho e^{-\alpha l}\underline{/-\theta}) \\ I &= I^+(e^{\alpha l}\underline{/\theta} - \rho e^{-\alpha l}\underline{/-\theta}) \end{aligned} \tag{1·129}$$

where l is the absolute value of the distance from the receiving end of the line to the point in question

Dividing V by I to obtain the input impedance, Z_{in}, we find that

$$Z_{in} = Z_0 \frac{e^{\alpha l}\underline{/\theta} + \rho e^{-\alpha l}\underline{/-\theta}}{e^{\alpha l}\underline{/\theta} - \rho e^{-\alpha l}\underline{/-\theta}} \tag{1·130}$$

Both numerator and denominator are multiplied by $e^{-\alpha l}$ to yield

$$Z_{in} = Z_0 \frac{1\underline{/\theta} + \rho e^{-2\alpha l}\underline{/-\theta}}{1\underline{/\theta} - \rho e^{-2\alpha l}\underline{/-\theta}} \tag{1·131}$$

Comparing Equation 1·131 with the Z_{in} of a dissipationless line as given by Equation 1·67, we find that they are the same except that ρ in Equation 1·131 is modified by the dissipation factor $e^{-2\alpha l}$. Thus the effective ρ of a line with dissipation varies with its length.

Let ρ_a be the reflection factor of a line with dissipation to be used for measuring input impedances. Thus

$$\rho_a = \rho e^{-2\alpha l} \tag{1·132}$$

Substituting Equation 1·32 into Equation 1·31, we find that

$$Z_{in} = Z_0 \frac{1\underline{/\theta} + \rho_a\underline{/-\theta}}{1\underline{/\theta} - \rho_a\underline{/-\theta}} \tag{1·133}$$

Equation 1·133 now has the same form as Equation 1·67 for the dissipationless line; but it employs the reflection factor as given by Equation 1·132. We see that the reflection factor decreases with the length l, showing that the standing wave ratio on a dissipative line decreases with distance from the receiving end. In fact, dissipative lines of sufficient length to have a ρ_a small enough to be negligible are used as load impedances which remain unchanged with frequency. ρ_a decreases rapidly with length of line because the wave has to travel from the point l to the end of the line, be reflected, and travel back again, thus suffering a double attenuation along the line.

All the calculations for the dissipationless line apply to the line with dissipation, except that ρ_a is used instead of ρ. Over a small region of the line, ρ_a can be assumed constant and the first approximation of a line stub calculated if necessary. The stubbing procedure outlined for the dissipationless case can then be used to obtain the final adjustments.

To calculate ρ_a, it is necessary to know the value of α. In the case of low loss lines,

$$\alpha = \frac{r}{2Z_0} \text{ nepers per unit length} \tag{1·134}$$

where r is the resistance of the line per unit length The value of r will vary with the frequency, f, being used

The value of r for the coaxial line shown in Figure 1.5a is given by

$$r = 8\ 3\sqrt{f}\left(\frac{1}{d} + \frac{1}{D}\right) \times 10^{-6} \text{ ohms per meter} \qquad (1\ 135)$$

where both d and D are measured in centimeters For the balanced transmission line shown in Figure 1 5b,

$$r = \frac{16\ 6\sqrt{f}}{d} \times 10^{-6} \text{ ohms per meter} \qquad (1\ 136)$$

where again d is measured in centimeters The resistance of the four-wire transmission line shown in Figure 1 5c is one half the resistance obtained in Equation 1 136 for the balanced line.

EXAMPLE 1 11 Determine the input impedance of an open-air balanced line made up of conductors 1 centimeter in diameter The line is 30 meters long and shorted at the far end The characteristic impedance of the line is 100 ohms and the frequency being used is 100 megacycles

Solving for r from Equation 1 136, we obtain

$$r = \frac{16\ 6\sqrt{10^8}}{1} \times 10^{-6} \text{ ohms per meter}$$

$$r = 0\ 166 \text{ ohm per meter}$$

Using Equation 1 134 to obtain α, we get

$$\alpha = \frac{0\ 166}{200}$$

$$\alpha = 0\ 00083 \text{ ohm per meter}$$

Solving for ρ from Equation 1 128, we find that

$$\rho = \frac{0 - 100}{0 + 100} = -1$$

Substituting into Equation 1 132, we obtain

$$\rho_a = -e^{-0\ 0498}$$

$$\rho_a = -0\ 951$$

Now Equation 1 133 may be used to obtain Z_{in} Remembering that in an open air line the electrical length is the same as the mechanical length, we obtain

$$Z_{in} = 100\frac{1\underline{/3600°} - 0\ 951\underline{/3600°}}{1\underline{/3600°} + 0\ 951\underline{/3600°}}$$

or

$$Z_{\text{in}} = 2.5 \text{ ohms} \qquad \textit{Ans.}$$

REFERENCE READING

E. A. Guillemin, *Communication Networks*, New York, John Wiley & Sons, 1935, Vol. II, Chapters 1, 2, and 3.

R. W. P. King, H. R. Mino, and A. H. Wing, *Transmission Lines, Antennas, and Wave Guides*, New York, McGraw-Hill Book Co., 1945, Chapter I.

S. A. Schelkunoff, *Electromagnetic Waves*, New York, D. Van Nostrand and Co., 1943, Chapter VII.

W. L. Everitt, *Communication Engineering*, New York, McGraw-Hill Book Co., pp. 94–178.

PROBLEMS

1·1 What is the input impedance of a dissipationless transmission line which has the following constants?

$$Z_0 = 100 \text{ ohms}$$

$$Z_L = (20 + j30) \text{ ohms}$$

$$\beta l = 75^\circ$$

1·2 Plot a curve of the input impedance magnitude and phase angle of problem 1·1 as βl varies from 0° to 360°.

1·3 What is the Z_0 of a transmission line that will match a 30-ohm resistive load to a line with a characteristic impedance of 140 ohms? Use a quarter-wave matching section.

1·4 Plot a curve of the effective inductance of a shorted dissipationless transmission line as its electrical length varies from 0° to 90°. The characteristic impedance of the line is 50 ohms and the frequency is 200 megacycles.

1·5 Plot a curve of the effective capacitance of the line of problem 1·4 when the load end is kept open.

1·6 Calculate the load impedance on a transmission line whose standing wave ratio is 3.5 and whose minimum occurs 12 centimeters closer to the load than when the load was shorted. The frequency is 200 megacycles and the dielectric is air. The Z_0 of the line is 80 ohms.

1·7 Calculate the load impedance on a transmission line where the standing wave ratio is infinite and the minimum occurs 65° closer to the generator than when the load was shorted. The Z_0 of the line is 100 ohms.

1·8 Prove that

where Z_{ss} is the impedance of a transmission line of length $\theta°$ with the load end shorted whereas Z_{so} is the impedance of the same transmission line with the load end open.

1·9 Calculate the length of stub and its position necessary to match the line of problem 1·3.

1·10 Calculate the length of stub and its position necessary to match the line of problem 1·7.

1·11 Given the two loads: (*a*) $(100 + j50)$ ohms; (*b*) $(30 + j70)$ ohms. Calculate the positions of all the stubs necessary to feed twice as much power to load *b* as is delivered to load *a*. Both are fed from a common generator using 100-ohm transmission lines. It is also desirable that the combination present an impedance of 100 ohms at the generator terminals.

Chapter 2

ELEMENTS OF VECTOR ANALYSIS

2·1 SCALAR AND VECTOR QUANTITIES

There are two fundamental types of physical quantities, scalars and vectors A scalar quantity has magnitude only. Examples of scalar quantities are mass, volume, density, and temperature. They will be represented by ordinary capital letters, such as M (for mass). A vector, for our purposes, may be defined as a quantity which has both magnitude and direction. Examples of vector quantities are velocity, force, electric field intensity, current density, and magnetic force.

The vector should not be confused with the representation of a sinusoidal function by means of the real part of a rotating phasor. The vector represents the actual instantaneous magnitude of the quantity and indicates its direction in three-dimensional space. The phasor, on the other hand, represents the magnitude variation of a sine function.

2·2 VECTOR DIAGRAMS

A vector acting at a point O may be represented by an arrow pointing in the direction of the vector, as shown in Figure 2·1. The magnitude of the vector is represented by the length of the line OA in the figure.

A vector is symbolized by bold-faced type, such as **M**. The magnitude of the vector will be represented by $|\mathbf{M}|$ or M. Notice that the magnitude of a vector is a scalar. An example of such a scalar is speed, where velocity is the vector possessing both magnitude and direction while speed is a scalar representing only the magnitude of the velocity.

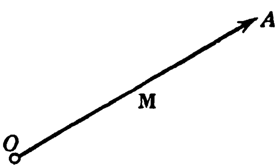


FIG. 2·1 Diagrammatic representation of a vector acting at the point O in the direction OA.

A vector diagram, such as shown in the figure, is usually drawn as a three-dimensional space diagram; but it represents a vector acting on a *single* point. Interaction between vectors, one requisite for vector calculations, can take place only if the vectors involved act on the same point. Thus, when two vectors of the same type act on the same point they may be combined; but when they act on different

points, the effects must be transferable to the same point before they are combined or even drawn in the normal manner in the same vector diagram.

Since the magnitude of a vector is a scalar, a scalar operating on a vector operates only on the magnitude, yielding another vector possessing the same directional characteristics as the original vector but differing in magnitude. Multiplying the vector **M** by the scalar l, we find that

$$l\mathbf{M} = \mathbf{Q} \tag{2·1}$$

where **Q** is the resultant vector having the same direction as **M**, but whose magnitude, Q, is equal to lM. This operation is often encountered when changing units. For instance, a vector representing veloc-

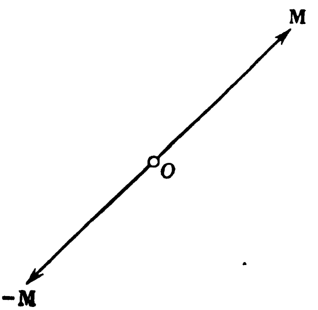


FIG. 2·2 The negative of a vector showing how it possesses the same magnitude but points in the opposite direction.

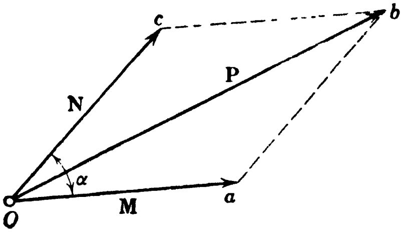


FIG. 2·3 The addition of two vectors where the sum is equal to the diagonal of the parallelogram whose two adjacent sides are the original vectors.

ity in feet per minute can be converted to a vector representing the velocity in inches per minute by multiplying with the conversion scalar 12. This increases the magnitude of the vector by a factor of 12 but has no effect on its direction.

In scalar mathematics, a negative quantity is a quantity less than zero. This is not so in vector calculations. The negative sign in vector notation possesses the ability to reverse the direction of a vector. In other words, the negative of a vector is another vector having the same magnitude as the original vector but acting in the opposite direction. In Figure 2·2 this axiom is illustrated by having the vector $-\mathbf{M}$ point in the opposite direction of the vector **M**.

2·3 ADDITION AND SUBTRACTION OF VECTORS

The addition of vectors must take into account both the directions and the magnitudes of the vectors concerned. It is carried out graphi-

cally as shown in Figure 2·3, where

$$\mathbf{M} + \mathbf{N} = \mathbf{P} \tag{2·2}$$

The addition is accomplished by completing the parallelogram formed by the two vectors, *ab* parallel to **N** and *cb* parallel to **M**. The result is a diagonal originating at the origin. Since both these vectors represent individual vector quantities acting on the point O, the sum also acts on the point O. Reversing the order of addition shown in Figure 2·2 will yield the same result; the commutative law of addition for scalars holds also for vectors. More than two vectors are added by first obtaining the sum of two vectors and then adding to that sum a third vector; the latter sum is then added to the fourth vector and so on until the complete sum is obtained. The order in which the vectors are taken has no effect on the result; commutative law of addition also applies when more than two vectors are added.

Subtraction is carried out by, first, reversing the direction of the vector following the minus sign and, then, carrying out addition. For instance,

$$\mathbf{P} - \mathbf{L} = \mathbf{R} \tag{2·3}$$

is the same as

$$\mathbf{P} + (-\mathbf{L}) = \mathbf{R} \tag{2·4}$$

where $-\mathbf{L}$ is a vector acting in the opposite direction to the vector **L**.

The components of a vector are any set of vectors whose sum is equal to the original vector. These components, of course, are usually chosen in a systematic manner so that they follow certain definite rules. The components most often used are called the rectangular components. They are three vectors parallel to the mutually perpendicular x, y, and z axes. The components of the vector **M**, for instance, are given by

$$\mathbf{M} = \mathbf{i}M_x + \mathbf{j}M_y + \mathbf{k}M_z \tag{2·5}$$

where normally, as shown, each component is represented as a product. The vectors **i**, **j**, and **k** are unit vectors having a magnitude of unity and pointing in the positive x, y, and z directions, respectively. M_x, M_y, and M_z are scalars which determine the component magnitudes. These scalars alone are often referred to as the components inasmuch as the subscript always denotes the unit vector to be used for determining the direction. As shown in Figure 2·4, the components of a vector are equal to its projections on the rectangular axes.

The application of components in the addition of vectors may be

illustrated by substituting into Equation 2·2 the components of the vectors involved:

$$\mathbf{M} + \mathbf{N} = \mathbf{i}M_x + \mathbf{j}M_y + \mathbf{k}M_z + \mathbf{i}N_x + \mathbf{j}N_y + \mathbf{k}N_z \tag{2·6}$$

In accordance with the process of addition, vectors pointing in the same direction may be added directly. Adding x, y, and z components of Equation 2·6, we obtain

$$\mathbf{M} + \mathbf{N} = \mathbf{i}(M_x+N_x) + \mathbf{j}(M_y+N_y) + \mathbf{k}(M_z + N_z) \tag{2·7}$$

This may be expressed by the statement: For any number of vectors, the components obtained by taking the sums of similar components are equal to the corresponding components of the vector sum. Similarly, for subtraction the difference of the components is used so that

$$\mathbf{M} - \mathbf{N} = \mathbf{i}(M_x - N_x) + \mathbf{j}(M_y - N_y) + \mathbf{k}(M_z - N_z) \tag{2·8}$$

Thus the vector difference is obtained by subtracting each of the components of one vector from a similar component of the other.

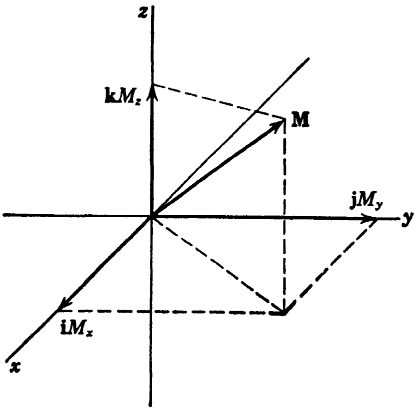


FIG. 2·4 The rectangular components of the vector **M**.

EXAMPLE 2·1 Find the sum and the difference of the two vectors

$$\mathbf{M} = \mathbf{i}8 + \mathbf{j}11 - \mathbf{k}3$$

$$\mathbf{N} = \mathbf{i}2 - \mathbf{j}9 + \mathbf{k}12$$

The sum, from Equation 2·7, is

$$\mathbf{M} + \mathbf{N} = \mathbf{i}(8 + 2) + \mathbf{j}(11 - 9) + \mathbf{k}(-3 + 12)$$

$$\mathbf{M} + \mathbf{N} = \mathbf{i}10 + \mathbf{j}2 + \mathbf{k}9 \qquad \textit{Ans. (a)}$$

The difference, from Equation 2·8, is

$$\mathbf{M} - \mathbf{N} = \mathbf{i}(8 - 2) + \mathbf{j}(11 + 9) + \mathbf{k}(-3 - 12)$$

$$\mathbf{M} - \mathbf{N} = \mathbf{i}6 + \mathbf{j}20 - \mathbf{k}15 \qquad \textit{Ans. (b)}$$

2·4 VECTOR OR CROSS PRODUCT

There are two types of multiplication involving vectors. The first type to be considered is called the vector or cross product; so-called because the result is a vector and the operation is signified by putting

a cross between the two vectors of the product. The vector product of the two vectors drawn in Figure 2·3 is noted as

$$\mathbf{M} \times \mathbf{N} = \mathbf{R} \tag{2·9}$$

The magnitude of the vector **R** is equal to the area of the parallelogram *Ocba* in the figure. Thus it is equal to $MN \sin \alpha$, where α is the angle between the two vectors. The direction of **R** is perpendicular to the plane of the two vectors pointing in the direction of travel of a right-

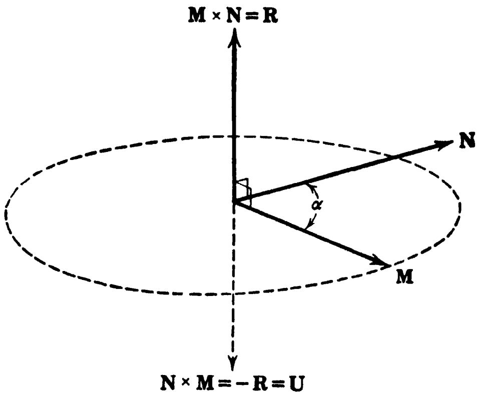


Fig. 2·5 The vector or cross product of two vectors where **R** is perpendicular to both **M** and **N**.

hand screw when the first vector of the product is rotated into the second through the smallest angle between them. Thus **R** in Figure 2·3 would be a vector perpendicular to the page surface pointing up out of the page. It is illustrated in Figure 2·5.

Reversing the order of the vectors reverses the direction of the result. Thus

$$\mathbf{N} \times \mathbf{M} = \mathbf{U} \tag{2·10}$$

where **U** in the figure points down, in the opposite direction of **R**, and therefore is equal to $-\mathbf{R}$. Hence reversing the order of the two vectors in a vector product changes the sign of the result.

To apply the vector product to the components of a vector, we must obtain the various vector products of the unit vectors **i**, **j**, and **k**. Taking all possible combinations of these vectors, we obtain

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} \\ \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} &= 0 \end{aligned} \tag{2·11}$$

As stated previously, reversing the order of any cross products like those in Equation 2·11 will place a negative sign in front of the result. An interesting point to note is that the cross product of a vector with itself is equal to zero. This is understandable because the area enclosed by completing the parallelogram would be zero.

To obtain the components of the cross product, $\mathbf{M} \times \mathbf{N}$ is expanded in terms of its components. It is accomplished by multiplying each of the components of one vector by the components of the other. Each result will involve a cross product of two unit vectors and a magnitude equal to the product of the magnitudes of the components. Thus

$$\begin{aligned} \mathbf{M} \times \mathbf{N} &= (\mathbf{i}M_x + \mathbf{j}M_y + \mathbf{k}M_z) \times (\mathbf{i}N_x + \mathbf{j}N_y + \mathbf{k}N_z) \\ &= \mathbf{i} \times \mathbf{i}M_xN_x + \mathbf{i} \times \mathbf{j}M_xN_y + \mathbf{i} \times \mathbf{k}M_xN_z \\ &+ \mathbf{j} \times \mathbf{i}M_yN_x + \mathbf{j} \times \mathbf{j}M_yN_y + \mathbf{j} \times \mathbf{k}M_yN_z \\ &+ \mathbf{k} \times \mathbf{i}M_zN_x + \mathbf{k} \times \mathbf{j}M_zN_y + \mathbf{k} \times \mathbf{k}M_zN_z \end{aligned} \qquad (2\cdot12)$$

Using Equation 2·11 so that all the cross products of similar vectors will drop out, and adding magnitudes multiplied by the same unit vector, we find that

$$\begin{aligned} \mathbf{M} \times \mathbf{N} &= \mathbf{i}(M_yN_z - M_zN_y) + \mathbf{j}(M_zN_x - M_xN_z) \\ &+ \mathbf{k}(M_xN_y - M_yN_x) \end{aligned} \qquad (2\cdot13)$$

The relationship in Equation 2·13 may be expressed as a determinant where the unit vectors occupy the first row; the components of the first vector, the second row; and the components of the second vector, the third row. With two vertical lines used to denote the determinant, it becomes

$$\mathbf{M} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M_x & M_y & M_z \\ N_x & N_y & N_z \end{vmatrix} \qquad (2\cdot14)$$

When the standard rules for the expansion of determinants is applied to Equation 2·14 the result is Equation 2·13, the expansion of the cross product.

Example 2·2 Determine the cross product $\mathbf{P} \times \mathbf{Q}$, where

$$\mathbf{P} = \mathbf{i}3 + \mathbf{j}5 + \mathbf{k}7$$

$$\mathbf{Q} = \mathbf{i}7 - \mathbf{j}2 + \mathbf{k}4$$

Substituting into Equation 2·13, using **P** instead of **M** and **Q** instead of **N**, we obtain

$$\mathbf{P} \times \mathbf{Q} = \mathbf{i}(20 + 14) + \mathbf{j}(49 - 12) + \mathbf{k}(-6 - 35)$$

$$\mathbf{P} \times \mathbf{Q} = \mathbf{i}34 + \mathbf{j}37 - \mathbf{k}41 \qquad \textit{Ans.}$$

2·5 Scalar or dot product

The other type of vector multiplication is called the scalar or dot product, so-called because the result is a scalar and the operation is signified by putting a dot between the two vectors being multiplied. The dot product of the two vectors shown in Figure 2·3 is represented as

$$\mathbf{M} \cdot \mathbf{N} = S = MN \cos \alpha \tag{2·15}$$

where α is the angle between the two vectors. In this case the result is a scalar which possesses only magnitude. Changing the order of the two vectors in scalar multiplication does not change the result. From the definition of a dot product we see that the dot product of a vector with itself is equal to a scalar having a magnitude equal to the vector's magnitude squared. Similarly, since the cosine term is involved, the dot product of two vectors at right angles to one another is zero. Applying these rules to the dot product of the unit vectors, we obtain

$$\begin{aligned} \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \\ \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0 \end{aligned} \tag{2·16}$$

To obtain the dot product of two vectors in terms of the vector components, we expand the dot product:

$$\begin{aligned} \mathbf{M} \cdot \mathbf{N} &= (\mathbf{i}M_x + \mathbf{j}M_y + \mathbf{k}M_z) \cdot (\mathbf{i}N_x + \mathbf{j}N_y + \mathbf{k}N_z) \\ &= \mathbf{i} \cdot \mathbf{i}M_xN_x + \mathbf{i} \cdot \mathbf{j}M_xN_y + \mathbf{i} \cdot \mathbf{k}M_xN_z \\ &+ \mathbf{j} \cdot \mathbf{i}M_yN_x + \mathbf{j} \cdot \mathbf{j}M_yN_y + \mathbf{j} \cdot \mathbf{k}M_yN_z \\ &+ \mathbf{k} \cdot \mathbf{i}M_zN_x + \mathbf{k} \cdot \mathbf{j}M_zN_y + \mathbf{k} \cdot \mathbf{k}M_zN_z \end{aligned} \tag{2·17}$$

Substituting the equalities of Equation 2·16 into Equation 2·17, we find that all the terms drop out except those involving the dot product of a unit vector with itself.

$$\mathbf{M} \cdot \mathbf{N} = M_xN_x + M_yN_y + M_zN_z \tag{2·18}$$

This discloses that the dot product of two vectors is a scalar equal to the sum of the products of the magnitudes of similar components.

EXAMPLE 2·3 Find the dot product $\mathbf{P} \cdot \mathbf{Q}$ of the vectors given in example 2·2.

Substituting into Equation 2·18, we obtain

$$\mathbf{P} \cdot \mathbf{Q} = 21 - 10 + 28$$

$$\mathbf{P} \cdot \mathbf{Q} = 39 \qquad \textit{Ans.}$$

2·6 SCALAR AND VECTOR FUNCTIONS OF POSITION IN SPACE

A function of position in space is a function having a definite value for each point in space. A scalar function of position in space has for each point in the spatial region a scalar magnitude; the function defines a scalar field. The density of the air above the earth is a scalar function of position in space. A numerical example of a scalar function of position in space is

$$A = 3x - 4y + 12z \tag{2·19}$$

This function, we see, has a definite value for each point of the coordinate system. If we add to this function another function of position, B, where

$$B = 5x + 15y - 10z \tag{2·20}$$

the terms add directly.

$$A + B = 3x + 5x + 15y - 4y + 12z - 10z \tag{2·21}$$

or

$$A + B = 8x + 11y + 2z \tag{2·22}$$

The scalar field may be a function of time which in itself is a function of the spatial coordinates.

$$D = [7x + 12y - z][\cos(\omega t - \beta y)] \tag{2·23}$$

The scalar D varies with the coordinates and also with time, the time factor (phase) being in itself a function of the y coordinate.

Similarly, when a vector has a definite value for each point in space, it is a vector function of position in space. The complete function defines a vector field. A magnetic field is an example of a vector function of position in space. A simple vector field is given by

$$\mathbf{P} = \mathbf{i}3x + \mathbf{j}4y - \mathbf{k}2z \tag{2·24}$$

Thus $\mathbf{P}$ is a vector with a definite value and direction at each point in space. Very often any of the components may be a function of all three coordinates. For example,

$$\mathbf{B} = \mathbf{i}(x + 3y - z) + \mathbf{j}\left(\frac{y^2 - z^2}{x}\right) + \mathbf{k}(x + y + z) \tag{2·25}$$

whereas **B** has a definite magnitude and direction for each point in space.

Since the vector field changes with position in space a vector diagram, which represents the vectors acting at one point, can present the vector field only at that point. However, complete fields can be manipulated mathematically by using the rules discussed in the previous paragraphs.

The vector fields can also be a function of time, such as

$$\mathbf{G} = \left[\mathbf{i}(x + 3y - z) + \mathbf{j}\left(\frac{y^2 - z^2}{x}\right) + \mathbf{k}(x + y + z)\right][\cos(\omega t - \beta x)] \tag{2·26}$$

G is a function of the coordinates as well as a function of the time t. Nearly all the vectors encountered in radiation are vector functions of position in space as well as vector functions of time.

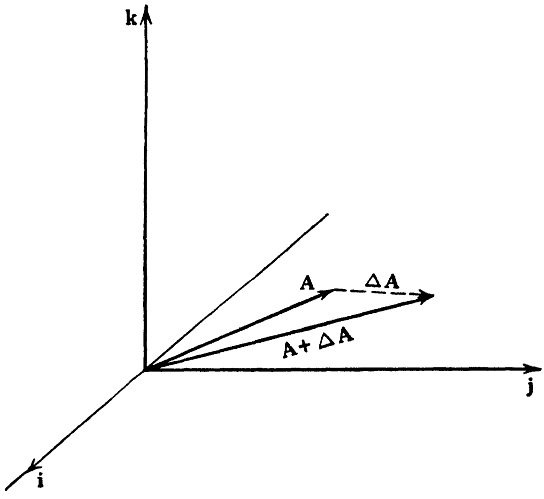


FIG. 2·6 The differential part of a vector where ΔA is the differential vector actually acting at the origin.

2·7 DIFFERENTIATION OF VECTORS

Differentiation in vector analysis is an extension of the scalar concept. When a variable in the expression for a vector changes a differential amount, the vector may change in both magnitude and direction. Thus the result of differentiating a vector will be a vector showing the changes in magnitude and direction of the original vector.

Let us consider the vector represented graphically in Figure 2·6:

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z \tag{2·27}$$

where A_x, A_y, and A_z are functions of t. Following the fundamental theory of differential calculus, we let t increase by Δt so that $\mathbf{A}$ will change by $\Delta\mathbf{A}$. When t changes by Δt, each of the components of $\mathbf{A}$ will also change by a related increment so that

$$\mathbf{A} + \Delta\mathbf{A} = \mathbf{i}(A_x + \Delta A_x) + \mathbf{j}(A_y + \Delta A_y) + \mathbf{k}(A_z + \Delta A_z) \qquad (2{\cdot}28)$$

To obtain the change in $\mathbf{A}$ it is necessary to subtract the original vector, as given by Equation 2·27, from the augmented vector, Equation 2·28, yielding

$$\Delta\mathbf{A} = \mathbf{i}(\Delta A_x) + \mathbf{j}(\Delta A_y) + \mathbf{k}(\Delta A_z) \qquad (2{\cdot}29)$$

Continuing the derivation, dividing both sides of Equation 2·29 by the change in the variable, Δt, and passing to the limit, we obtain

$$\frac{d\mathbf{A}}{dt} = \mathbf{i}\frac{dA_x}{dt} + \mathbf{j}\frac{dA_y}{dt} + \mathbf{k}\frac{dA_z}{dt} \qquad (2{\cdot}30)$$

This reveals that the derivative of a vector is equal to the vector sum of derivatives of its components. Since each rectangular component is constant in direction, the derivative of a rectangular component normally has the same direction as the component. The foregoing derivation uses the variable t but it may be repeated for any variable and will yield a similar result.

EXAMPLE 2·4 Differentiate, with respect to t, the vector

$$\mathbf{A} = \mathbf{i}(3t^2 - t) + \mathbf{j}(5t^2 - 7) + \mathbf{k}(4t - 1)$$

In this vector

$$A_x = 3t^2 - t$$

$$A_y = 5t^2 - 7$$

$$A_z = 4t - 1$$

so that

$$\frac{dA_x}{dt} = 6t - 1$$

$$\frac{dA_y}{dt} = 10t$$

$$\frac{dA_z}{dt} = 4$$

Substituting into Equation 2·30, we obtain

$$\frac{d\mathbf{A}}{dt} = \mathbf{i}(6t - 1) + \mathbf{j}10t + \mathbf{k}4 \qquad \textit{Ans}$$

2·8 Differentiation of vector functions

One of the simplest types of vector functions is the product of a scalar function and a vector, both of which are functions of the same variable. The derivative of $(a\mathbf{A})$, where both a and $\mathbf{A}$ are functions of the variable u, is obtained by following Equation 2·30.

$$\frac{d}{du}(a\mathbf{A}) = \mathbf{i}\frac{d}{du}(aA_x) + \mathbf{j}\frac{d}{du}(aA_y) + \mathbf{k}\frac{d}{du}(aA_z) \qquad (2·31)$$

Each of the differentiations indicated on the right-hand side of Equation 2·31 is an ordinary differentiation of the product of two functions of the same variable. Carrying out the differentiation, we get

$$\begin{aligned}\frac{d}{du}(a\mathbf{A}) &= \mathbf{i}\left(A_x\frac{da}{du} + a\frac{dA_x}{du}\right) \\ &\quad + \mathbf{j}\left(A_y\frac{da}{du} + a\frac{dA_y}{du}\right) + \mathbf{k}\left(A_z\frac{da}{du} + a\frac{dA_z}{du}\right)\end{aligned} \qquad (2·32)$$

Arranging the terms so that da/du and a can be factored out, we obtain

$$\frac{d}{du}(a\mathbf{A}) = (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)\frac{da}{du} + a\left(\mathbf{i}\frac{dA_x}{du} + \mathbf{j}\frac{dA_y}{du} + \mathbf{k}\frac{dA_z}{du}\right) \qquad (2·33)$$

Substituting the vector and the derivative of the vector for their components, we find that

$$\frac{d}{du}(a\mathbf{A}) = \mathbf{A}\frac{da}{du} + a\frac{d\mathbf{A}}{du} \qquad (2·34)$$

Equation 2·34 is similar to the equation for the differentiation of the product of two scalar functions. This, however, represents only one case. No rules should be transposed from scalar to vector calculus but each case of vector analysis differentiation should be worked out and rules obtained in that manner.

The method indicated above will yield the following results for scalar and vector products:

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{dt} + \mathbf{B}\cdot\frac{d\mathbf{A}}{dt} \qquad (2·35)$$

$$\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\times\mathbf{B} \qquad (2·36)$$

In the differentiation of the vector or cross product, we must take care that the order of the vectors in the products are not changed since

the sign of the result depends on the order. We see that in Equation 2·36 the order in both terms on the right-hand side of the equal sign is the same as the original order of the vectors.

2·9 Integration of vectors

Integration, as in scalar analysis, is the opposite of differentiation. Hence, since the differential of a vector is equal to the sum of the differentials of its components, conversely, the integral of a vector is the sum of the integrals of its components. Thus

$$\int \mathbf{B}\,du = \int (\mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z)\,du \tag{2·37}$$

where B_x, B_y, and B_z are all functions of u. Applying the rule of integration, we obtain

$$\int \mathbf{B}\,du = \mathbf{i}\int B_x\,du + \mathbf{j}\int B_y\,du + \mathbf{k}\int B_z\,du \tag{2·38}$$

where each integration is performed following the regular rules of integral calculus. Similarly, for the definite integral of a vector the result is the sum of the definite integrals of its components.

Example 2·5 Integrate with respect to the variable v the vector

$$\mathbf{D} = \mathbf{i}(4v - 7) + \mathbf{j}(12v^2 - 8v) + \mathbf{k}7$$

Using Equation 2·36, we obtain

$$D_x = 4v - 7$$

$$D_y = 12v^2 - 8v$$

$$D_z = 7$$

Remembering that there is a constant of integration that must be included, we find that

$$\int D_x\,dv = \int (4v - 7)\,dv = 2v^2 - 7v + C_x$$

$$\int D_y\,dv = \int (12v^2 - 8v)\,dv = 4v^3 - 4v^2 + C_y$$

$$\int D_z\,dv = \int 7dv = 7v + C_z$$

where C_x, C_y, and C_z are the respective integration constants. The sum of these integrals as components of the resultant vector is

$$\int \mathbf{D}\,dv = \mathbf{i}(2v^2 - 7v + C_x) + \mathbf{j}(4v^3 - 4v^2 + C_y) + \mathbf{k}(7v + C_z)$$

or

$$\int \mathbf{D}\,dv = \mathbf{i}(2v^2 - 7v) + \mathbf{j}(4v^3 - 4v^2) + \mathbf{k}(7v) + \mathbf{i}C_x + \mathbf{j}C_y + \mathbf{k}C_z$$

Now if

$$\mathbf{C} = \mathbf{i}C_x + \mathbf{j}C_y + \mathbf{k}C_z$$

then, as the final result,

$$\int \mathbf{D}\,dv = \mathbf{i}(2v^2 - 7v) + \mathbf{j}(4v^3 - 4v^2) + \mathbf{k}7v + \mathbf{C} \qquad \textit{Ans.}$$

This illustrates an interesting point. The constant of integration of a vector is also a vector.

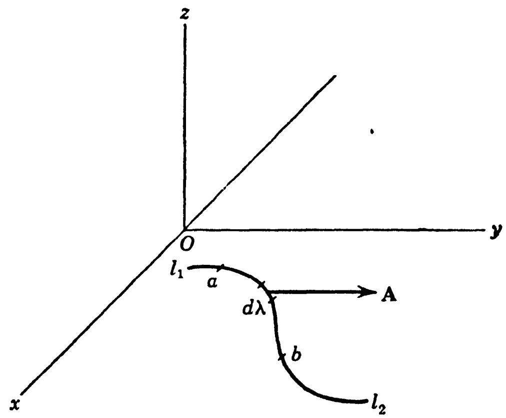


FIG. 2·7 The line integral of **A** along the line l_1l_2, where each differential part is equal to the dot product of **A** and $d\boldsymbol{\lambda}$.

2·10 LINE AND SURFACE INTEGRALS

Figure 2·7 shows a vector **A** which is a function of the coordinates. Also illustrated is a line l_1l_2 in the coordinate space. The integral of the tangential component of **A** along the line l_1l_2 is called the line integral of the vector **A**. The line integral is usually taken between two points on the line as follows:

$$\text{The line integral of } \mathbf{A} \text{ between } a \text{ and } b = \int_a^b \mathbf{A} \cdot d\boldsymbol{\lambda}$$

$$= \int_a^b A_x\,d\lambda_x + \int_a^b A_y\,d\lambda_y + \int_a^b A_z\,d\lambda_z \qquad (2{\cdot}39)$$

where $d\boldsymbol{\lambda}$ is an infinitesimal part of l_1l_2 pointing in the direction, along

the line, from a to b. If the path is a closed curve, the integral is written

$$\oint_l \mathbf{A} \cdot d\boldsymbol{\lambda} = \oint_l A_x \, d\lambda_x + \oint_l A_y \, d\lambda_y + \oint_l A_z \, d\lambda_z \tag{2·40}$$

Since the vector involved is a function of the coordinates, the integral depends on the exact path that is followed and, in the general case of a closed path, it does not necessarily vanish. A very common line integral is electromotive force, where

$$E = \int_a^b \mathbf{E} \cdot d\boldsymbol{\lambda} \tag{2·41}$$

in which E is the work done on a unit positive charge moving from a to b through an electric field of intensity $\mathbf{E}$.

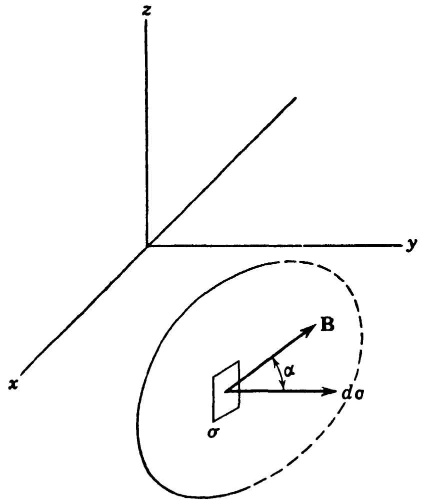


Fig. 2·8 The surface integral of $\mathbf{B}$ over the surface σ, where $d\sigma$ is an infinitesimal part of the surface σ.

In Figure 2·8 is a vector $\mathbf{B}$, which is a function of the coordinates. $\boldsymbol{\sigma}$ is a surface bounded by a line as shown. The integral of the normal component of the vector $\mathbf{B}$ over the surface $\boldsymbol{\sigma}$ is called the surface integral. It is represented as

$$\iint_\sigma \mathbf{B} \cdot d\boldsymbol{\sigma} = \iint B_x \, d\sigma_x + \iint B_y \, d\sigma_y + \iint B_z \, d\sigma_z \tag{2·42}$$

Thus a surface is represented by a vector normal to the surface and equal in magnitude to the area of the surface. If the surface is curved, it can be broken up into the sum of a series of vectors, each normal to a small segment of the surface.

The integral given in Equation 2·42 is also called the flux, ϕ, through the surface, $\boldsymbol{\sigma}$, where $\mathbf{B}$ is the flux density. If $\boldsymbol{\sigma}$ is a flat surface and $\mathbf{B}$ is constant over the surface,

$$\iint \mathbf{B} \cdot d\boldsymbol{\sigma} = \mathbf{B} \cdot \boldsymbol{\sigma} = B\sigma \cos \alpha \tag{2·43}$$

where B is the magnitude of $\mathbf{B}$; σ, the magnitude of $\boldsymbol{\sigma}$; and α, the angle between $\mathbf{B}$ and $\boldsymbol{\sigma}$.

2·11 Divergence of a vector

The divergence of a vector is just what the name implies. It is the outward flux, or flow, of a vector per unit volume as the volume approaches zero. For instance, a charged sphere would have a divergence inasmuch as the electric-field-intensity vector would spread outward from that point. When the charge is positive, the divergence is positive; and when the charge is negative, the flux points into the volume and the divergence is negative.

The divergence may be defined as the limit of the closed surface integral of a vector field, divided by the volume contained within this surface, the limit being obtained by allowing the volume to approach zero:

$$\text{divergence } \mathbf{A} = \underset{\Delta\tau \to 0}{\text{limit}} \frac{\oint\oint \mathbf{A} \cdot d\boldsymbol{\sigma}}{\Delta\tau} \tag{2·44}$$

The circles on the double integral sign indicate a closed surface. The volume $\Delta\tau$ is the volume contained within the surface $\boldsymbol{\sigma}$. The divergence of a vector is a magnitude only and is, therefore, a scalar. The equations for the divergence of the electric and magnetic quantities will be derived in the chapter on fundamental electromagnetic equations.

2·12 Curl of a vector

Curl, again, is what its name implies. It is a measure of the twist that can be caused by a vector field. For instance, let us consider a river where the water is flowing very slowly at the sides and very rapidly in the center. If a horizontal paddle wheel is placed in the river so that one edge just projects into the center of the river and the other edge lies in the slow flow at the side, the wheel will turn. It will turn with the greatest force about an axis perpendicular to the surface of the river. Thus it is said that the vector field of the velocity of the water has curl. A more emphatic indication of curl would be a whirlpool in the water since it would definitely produce circular motion in a paddle wheel. Inasmuch as the paddle wheel has to be immersed with its axis along a definite line to obtain the maximum rotary force, curl will have to be a vector having direction as well as magnitude, to be completely defined. The direction of the vector is given as the direction of the advance of a right-handed screw turning with the paddle wheel and whose axis is the axis of the paddle wheel.

The mathematical definition of a curl is in terms of its components.

The component of a curl, in a direction normal to the positive side of a surface, Δs, is given as the limit of the closed line integral of the vector field around a surface divided by the area of the surface as the surface is allowed to approach zero.

$$\left| \text{curl } \mathbf{A} \right|_{\substack{\text{component} \\ \text{perpendicular} \\ \text{to } s}} = \lim_{\Delta s \to 0} \frac{\oint_l \mathbf{A} \cdot d\boldsymbol{\lambda}}{\Delta s} \tag{2·45}$$

where l is the line around the surface Δs. To define the curl completely, we should obtain all the necessary components at the point in question and then add them vectorially, each component being multiplied by its proper unit vector. The equations for the curl of the electric and magnetic field intensities will be derived in the chapter on fundamental electromagnetic equations.

REFERENCE READING

L. Page and N. I. Adams, *Electrodynamics*, New York, D. Van Nostrand Co., 1940, Chapter I.

S. A. Schelkunoff, *Electromagnetic Waves*, New York, D. Van Nostrand Co., 1943, Chapter I.

H. H. Skilling, *Fundamentals of Electric Waves*, New York, John Wiley & Sons, 1942, Chapter II.

Useful vector equations

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{2·46}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{2·47}$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{2·48}$$

$$\mathbf{A} \times \mathbf{A} = 0 \tag{2·49}$$

$$\mathbf{A} \cdot \mathbf{A} = A^2 \tag{2·50}$$

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{D} = \mathbf{A} \times \mathbf{D} + \mathbf{B} \times \mathbf{D} \tag{2·51}$$

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{D} = \mathbf{A} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{D} \tag{2·52}$$

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{D} = -\mathbf{B} \times \mathbf{A} \cdot \mathbf{D} = \mathbf{D} \cdot \mathbf{A} \times \mathbf{B} \tag{2·53}$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{B} = (\mathbf{A} \cdot \mathbf{B})\mathbf{B} - B^2\mathbf{A} \tag{2·54}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{D})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{D} \tag{2·55}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{D} \times \mathbf{E}) = (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{E}) - (\mathbf{A} \cdot \mathbf{E})(\mathbf{B} \cdot \mathbf{D}) \tag{2·56}$$

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{D} \times \mathbf{E}) = (\mathbf{A} \times \mathbf{B} \cdot \mathbf{E})\mathbf{D} + (\mathbf{B} \times \mathbf{A} \cdot \mathbf{D})\mathbf{E} \tag{2·57}$$

PROBLEMS

2·1 Find the sum of the vectors

$$\mathbf{A} = \mathbf{i}4 + \mathbf{j}7 - \mathbf{k}8$$

$$\mathbf{B} = \mathbf{i}7 + \mathbf{j}12 + \mathbf{k}9$$

$$\mathbf{C} = \mathbf{i}10 + \mathbf{j}4 + \mathbf{k}3$$

2·2 Find the dot product of $\mathbf{A} \cdot \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ have the values given in problem 2·1.

2·3 Find the vector product $\mathbf{B} \times \mathbf{C}$, where $\mathbf{B}$ and $\mathbf{C}$ have the values given in problem 2·1.

2·4 Prove Equation 2·53.

2·5 Differentiate with respect to t

$$\mathbf{E} = \mathbf{i}(3t^2 - 4x + z) + \mathbf{j}(2xt - yt^2) + \mathbf{k}(4t^2 + xt)$$

2·6 Integrate $\mathbf{E}$ of problem 2·5 with respect to t.

Chapter 3

FUNDAMENTAL ELECTROMAGNETIC EQUATIONS

3·1 Introduction

In formulating the fundamental electromagnetic equations, the rationalized meter-kilogram-second-coulomb system of units, commonly known as the M.K.S. system, will be used throughout. In this system the stated quantities are related to one another in practical units of ohm, ampere, volt, mho, and so on. In all, by employing these units, we avoid confusion when the equations are used for practical applications.

The experimental evidence obtained, so far, tends to show that electromagnetic phenomena are governed by the fundamental equations first postulated by Maxwell about seventy-five years ago in his *Treatise on Electricity and Magnetism.* Maxwell's equations are used as the fundamental equations in all electromagnetic problems encountered in this book. We assume that the reader has a general knowledge of electricity and magnetism such as may be obtained from a course in basic physics. What Maxwell's equations do is to reduce the basic laws of electricity and magnetism to a set of point functions, combine them, and yield a set of expressions for the complete electromagnetic field.

Their application, in a great many cases, follows a simple rule. First, Maxwell's equations are set down in the most adaptable form for the problem. Second, the boundary conditions of the problem are inserted. Third, the resultant equations are solved.

The tremendous power behind the use of Maxwell's equations in solving electromagnetic problems will become apparent to the reader as the equations are applied. This chapter will be devoted to the formulation of Maxwell's equations with a view to arriving at a physical picture of what they mean. No numerical problems are given in this chapter since it deals only with the basic fundamental electromagnetic relationships. The numerical problems are taken up in the succeeding chapters.

3·2 Units

Electric intensity is the force that is exerted on a unit positive charge of electricity at a specific point, provided that the unit charge can be introduced at that point without changing the electric intensity. A

field of points of electric intensity is called an electric field. The electric intensity, since it is a force and therefore has both magnitude and direction, is a vector which will be designated by **E**, where

$$\mathbf{E} = \mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z \tag{3·1}$$

E_x, E_y, and E_z are the scalar components of **E** in the direction of the cartesian coordinate axes noted as the x, y, and z axes. The magnitude of **E**, designated by E, is given by

$$E = |\mathbf{E}| = \sqrt{E_x^2 + E_y^2 + E_z^2} \tag{3·2}$$

which is the customary equation for the magnitude of a vector in terms of its rectangular components.

The electric induction at any point is also a vector, designated by **D**, where

$$\mathbf{D} = k_e\epsilon_0\mathbf{E} \tag{3·3}$$

The term k_e is the relative dielectric constant of the medium with respect to empty space. It is a dimensionless constant and is the value usually given in tables. It is necessary to use ϵ_0 so that the relative dielectric constants, given in nearly all available tables, can be used directly in the resultant equations. The value of ϵ_0 in the M.K.S. system is given by

$$\epsilon_0 \simeq \frac{1}{36\pi} \times 10^{-9} \simeq 8.85 \times 10^{-12} \text{ farad per meter} \tag{3·4}$$

In a magnetic field the magnetic intensity at any point is the force that would be exerted on a unit north pole—if such a pole were available—provided the pole did not affect the magnetic field. Since the magnetic field has both magnitude and direction, it too is a vector and is noted as **H**, where

$$\mathbf{H} = \mathbf{i}H_x + \mathbf{j}H_y + \mathbf{k}H_z \tag{3·5}$$

The magnetic induction, at a point in a medium, is the product of the magnetic intensity and the permeability of the medium at that point. It is noted as **B**, where

$$\mathbf{B} = k_m\mu_0\mathbf{H} \tag{3·6}$$

k_m is a dimensionless constant known as the relative permeability of the medium with respect to free space. It is the constant available in nearly all the tables in common use. The term μ_0 is determined by the M.K.S. system of units and has the value given by

$$\mu_0 \simeq 4\pi \times 10^{-7} \simeq 1.26 \times 10^{-6} \text{ henry per meter} \tag{3·7}$$

The total magnetic lines of flux, ϕ, is the name given to the number of lines of magnetic induction cut by the surface, s, under consideration. Thus it is dependent on the area of the surface, the intensity of the magnetic induction, and the angle between the lines of magnetic induction and the surface. The flux may be obtained by taking the dot product of the magnetic induction, **B**, and the vector differential surface, $d\mathbf{s}$. To obtain the total flux, it is neces ary to integrate over the surface, s. Thus the definition for ϕ is given by

$$\phi = \iint_s \mathbf{B} \cdot d\mathbf{s} \tag{3·8}$$

Similarly, the total electric flux, ψ, through a surface, s, is defined as the total number of lines of electric induction cut by the surface under consideration. Again the dot product is used, but this time the dot product of the electric induction and the differential area is employed. Thus ψ is expressed by

$$\psi = \iint_s \mathbf{D} \cdot d\mathbf{s} \tag{3·9}$$

The density of electric charge in a medium is noted as ρ. The total charge, q, in a volume, τ, is equal to the volume integral of $\rho\, d\tau$ taken over the volume τ. Hence the equation for q is

$$q = \iiint_\tau \rho\, d\tau \tag{3·10}$$

All intensities and densities are usually stated in terms of per unit area or per unit volume. The intensity or density for a differential area or volume is expressed in the same terms but does not have to extend further than the space under consideration. In other words, the density of charge for a differential volume is that density per unit volume which, when distributed evenly throughout a unit volume, will give to each differential of that unit volume an equivalent charge.

3·3 The Divergence of D, B and I

Gauss's law states that the total electric flux, ψ, emanating from a closed surface (defined by Equation 3·9 when the integral is taken over a closed surface) is equal to the charge, q, enclosed within the surface. The electric induction lines pointing out through the surface are positive, and those pointing into the surface are negative. If there are more of either type there must be a charge within the surface that the extra lines terminate upon. Thus

$$\psi = \oint\!\!\oint \mathbf{D} \cdot d\mathbf{s} = q \tag{3·11}$$

To obtain the expression for the divergence of **D**, Gauss's law is reduced to a point relationship. In Figure 3·1 is shown an infinitesimal volume, $\Delta\tau$, in rectilinear coordinates. This infinitesimal volume can be expressed in terms of the x, y, and z coordinates by

$$\Delta\tau = \Delta x \, \Delta y \, \Delta z \tag{3·12}$$

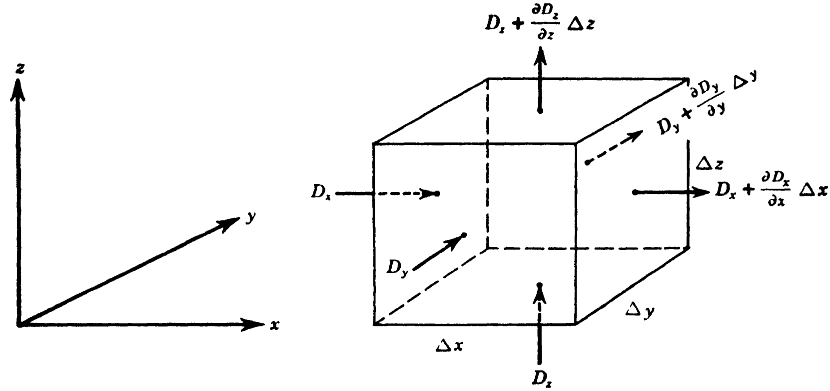


FIG. 3·1 An infinitesimal volume $\Delta x \, \Delta y \, \Delta z$, showing the average values of the components of **D** acting on the six sides of the closed surface.

Let us assume that there is an electric induction **D** existing in the space. This electric induction may vary throughout the coordinate space. Over each of the surfaces of the infinitesimal volume, however, each of the components of **D** will have an average value, as indicated in the figure. For example, D_y is the average value of the y component of **D** on the positive side of the volume over the surface $\Delta z \, \Delta x$. By means of Taylor's theorem and (since infinitesimal distances are involved) neglecting second-order differentials, the average value of the y component of **D** on the opposite side of the volume is given by

$$D_y + \frac{\partial D_y}{\partial y} \Delta y \tag{3·13}$$

The boundaries of the volume are so chosen that only one component of electric induction penetrates each of the sides. The total number of lines, $\Delta\psi$, over the surface, s, of the infinitesimal volume, $\Delta\tau$, can be obtained by multiplying each of the areas of the surface by the average value of the electric induction through that surface. All the average values are obtained in a

obtained. Carrying out this operation, we obtain

$$\begin{aligned} \Delta\psi = &-D_x\,\Delta y\,\Delta z + \left(D_x + \frac{\partial D_x}{\partial x}\,\Delta x\right)\Delta y\,\Delta z \\ &-D_y\,\Delta z\,\Delta x + \left(D_y + \frac{\partial D_y}{\partial y}\,\Delta y\right)\Delta z\,\Delta x \\ &-D_z\,\Delta x\,\Delta y + \left(D_z + \frac{\partial D_z}{\partial z}\,\Delta z\right)\Delta x\,\Delta y \end{aligned} \qquad (3\cdot14)$$

Again, those vectors pointing outward from the volume are called positive, and those pointing into the volume are called negative. Equation 3·14 can be simplified to

$$\Delta\psi = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{dz}\right)\Delta x\,\Delta y\,\Delta z \qquad (3\cdot15)$$

From Equation 3·12, $\Delta\tau$ can be substituted for $\Delta x\ \Delta y\ \Delta z$. Also, from Gauss's law, stated in Equation 3·11, $\Delta\psi$ is equal to the charge, q, contained in the volume $\Delta\tau$.

$$q = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta\tau \qquad (3\cdot16)$$

From Equation 3·11, q is equal to the surface integral of the electric induction. Inasmuch as q is the total charge enclosed within the surface, it is also equal to the average charge density, ρ, multiplied by the volume $\Delta\tau$. Substituting for q in Equation 3·16, we find that

$$\oint\!\!\oint_s \mathbf{D}\cdot d\mathbf{s} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta\tau = \rho\Delta\tau \qquad (3\cdot17)$$

Each of these three equalities can be divided by the infinitesimal volume and still maintain the equality.

$$\frac{\oint\!\!\oint_s \mathbf{D}\cdot d\mathbf{s}}{\Delta\tau} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho \qquad (3\cdot18)$$

Equation 3·18 states that the surface integral of the electric induction over the surface of a volume, $\Delta\tau$, divided by that volume, is equal to the sum of the components of the derivative of the electric induction which, in turn, is equal to the average charge density within the volume. However, from Equation 2·44 in Chapter 2, the divergence of a vector field is defined as the limit of the closed surface integral of the vector divided by the volume contained within this surface as the volume is

allowed to approach zero. Thus the divergence of **D** is given by

$$\text{divergence } \mathbf{D} = \underset{\Delta\tau \to 0}{\text{limit}} \left(\frac{\oint\oint_s \mathbf{D} \cdot d\mathbf{s}}{\Delta\tau} \right) \tag{3·19}$$

This can now be applied to the equation for the surface integral of the electric induction, Equation 3·18. However, when the volume is allowed to approach zero, all the average values assumed become the actual values at the point about which the volume approaches zero. Combining Equation 3·18 and 3·19, we obtain the expression for the divergence of the electric induction, **D**, about a point:

$$\text{divergence } \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho \tag{3·20}$$

where D_x, D_y, and D_z are the components of the vector **D** at the point under consideration, and ρ is the charge density at the same point. Thus in rectangular coordinates the divergence of **D** is equal to the sum of the components of the derivative of the vector **D**, which in turn is equal to the charge density at the point in question.

Consider now the meaning of Equation 3·20. For the sum of the components of the derivative of **D** not to vanish, the number of lines of electric induction pointing into the infinitesimal volume, at the point under consideration, has to differ from the number of lines pointing out of the infinitesimal volume. Since all lines have to terminate on a charge, there must be a certain charge density at the point for the lines to terminate upon. This charge density will be proportional to the difference in the number of lines which, in turn, will be equal to the sum of the components of the derivative of **D**. As we can see from Equation 3·20, in the M.K.S. system it is directly equal to that sum.

The symbol ∇, called either del or nabla, is used to represent a vector operator where

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{3·21}$$

Thus ∇ is an operator which standing by itself has no meaning or value. It resembles a vector whose components are partial derivative operators. However, these components are only effective when there is something on which to operate. It is usually used in vector equations like any other vector; but in the final result, when one of the components of nabla appears multiplying a vector (or component of a vector), it signifies the partial derivative of the multiplied quantity.

The dot product of nabla and the vector **D** will yield the sum of the components of the derivative of **D**, or, in other words—Equation 3·20—the divergence of **D**:

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \text{divergence } \mathbf{D} \tag{3·22}$$

It can be shown that in any coordinate system the dot product of nabla and a vector is equal to the divergence of the vector. Equation 3·20 can now be written as

$$\nabla \cdot \mathbf{D} = \rho \tag{3·23}$$

Thus the divergence of the electric induction vector **D** at a point is equal to the charge density at that point.

The same procedure may be followed using the magnetic induction **B**. In all practical applications, however, there is never a unit north magnetic pole without a corresponding unit south magnetic pole. Thus, in practice, the lines of magnetic flux do not terminate, being always closed loops. At any point, then, there will always be as many magnetic lines going into an infinitesimal volume surrounding the point as there are going out. The sum of the components of the derivative of the magnetic induction will therefore be zero, or

$$\nabla \cdot \mathbf{B} = 0 \tag{3·24}$$

Equation 3·24 states that within a medium the divergence of the magnetic induction, **B**, is always equal to zero; or within a medium the lines of magnetic induction are everywhere continuous

To obtain the divergence of current, let us assume a coordinate space with current flowing through it. Any volume, $\Delta\tau$, will have a total current flowing away from it, I, which is equal to the rate of change of the charge, q, enclosed within the surface:

$$I = \frac{dq}{dt} \tag{3·25}$$

The charge q, however, is equal to the average charge density, ρ, times the volume $\Delta\tau$. Since the volume is a constant, substituting this relationship into Equation 3·25 will give

$$I = \Delta\tau \frac{\partial \rho}{\partial t} \tag{3·26}$$

This states that the current flowing out of an infinitesimal volume, $\Delta\tau$, is equal to the product of the volume and the partial derivative with respect to time of the average charge density. Equation 3·26 can now be used in deriving the equation for the divergence of the current

density vector field, **I**. **I** is a vector equal in magnitude to the current density flowing at any point in the field and having the direction of the current flow. The divergence of the current density can now be derived in a manner similar to the derivation of the divergence of the electric induction. The result is

$$\nabla \cdot \mathbf{I} = \frac{\partial \rho}{\partial t} \tag{3·27}$$

Since, in deriving the divergence, the volume $\Delta\tau$ is allowed to approach zero, the charge density ρ becomes the charge density at the point under consideration. Thus the divergence of the current density at a point is equal to the derivative of the charge density at the point. (*Note*: The symbol **I** for current density must not be confused with the symbol **i** for the unit vector in the positive x direction.)

Once the idea of divergence becomes clear as signifying the lines of a vector force spreading out from a point, the equations for divergence become rather self-evident. The divergence is always a scalar and can be readily recognized as such, being equal to the dot product of nabla and a vector.

3·4 The curl of H and E

The magnetomotive force between two points is defined as the work done in carrying a unit north pole, if such a pole existed, between the two points. In a magnetic field, **H**, the magnetomotive force, mmf, between the two points A and B is equal to the line integral of the vector **H** between the two points.

$$\int_A^B \mathbf{H} \cdot d\mathbf{l} = \int_A^B H \cos\theta \, dl = \text{mmf} \tag{3·28}$$

where H is the magnitude of **H**; $d\mathbf{l}$, a differential length of line between A and B; and θ, the angle between **H** and $d\mathbf{l}$—the direction of $d\mathbf{l}$ being the direction it would be traversed when moving from A to B along the line. The mmf equation states that when a unit north pole is carried around a closed path the mmf is equal to the current enclosed within the path. The current enclosed is defined as the total current crossing any surface the closed path outlines. Since the value of current is unique, no matter which of the infinite possible surfaces is chosen, the value of total current obtained will always be the same. Calling the closed path line l, we can write

$$\oint_l \mathbf{H} \cdot d\mathbf{l} = I \tag{3·29}$$

where I is the total current enclosed within the closed path.

The curl of **H** is obtained when Equation 3·29 is reduced to a point relationship. To do this, let us consider the spatial relationship where a current density, **I**, is flowing and a vector field, **H**, is present. In Figure 3·2 is shown, in rectangular coordinates, an infinitesimal surface $\Delta y\ \Delta z$. This surface is located in the x plane and the sides of the

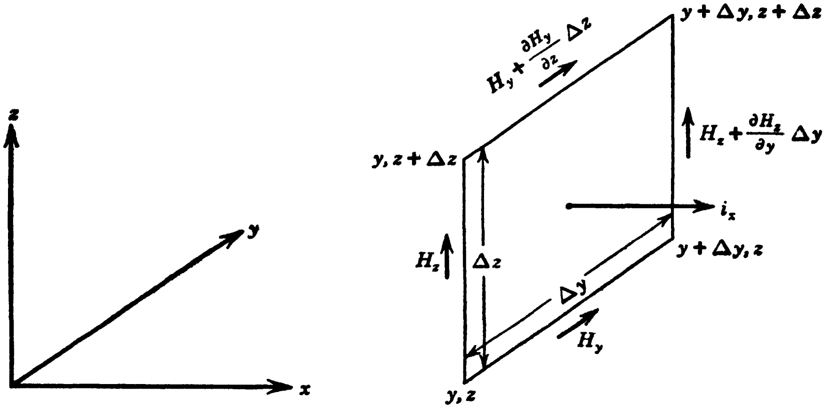


FIG. 3·2 An infinitesimal surface $\Delta y\ \Delta z$, with an average value of current i_x flowing normal to the surface. The average values of the components of **H** are also shown.

infinitesimal area are chosen parallel to the coordinate axes. The component of **I** flowing through the surface is i_x. H_z is the average value of **H** acting along the side of the area from y, z to y, $z + \Delta z$. By means of Taylor's theorem, and since second-order differentials can be neglected for infinitesimal distances, the average value of **H** on the opposite side of the area acting from $y + \Delta y$, z to $y + \Delta y$, $z + \Delta z$ is

$$H_z + \frac{\partial H_z}{\partial y} \Delta y \tag{3·30}$$

Similarly, the average value of **H** from y, z to $y + \Delta y$, z is taken as H_y so that the average value of **H** along the opposite side from y, $z + \Delta z$ to $y + \Delta y$, $z + \Delta z$ is

$$H_y + \frac{\partial H_y}{\partial z} \Delta z \tag{3·31}$$

The next step is to take the line integral of **H** around the area $\Delta y\ \Delta z$. The counterclockwise direction is positive in the figure since it conforms with the direction of rotation of a right-handed screw advancing in the direction of i_x. The **H** vector components involved are parallel to the sides of the area; thus the dot product for each side is equal to

the product of the magnitude of the **H** vector component, pointing in the direction of the side and the length of the side. The closed line integral around the area is given by

$$\oint_l \mathbf{H} \cdot d\mathbf{l} = -H_z\,\Delta z + H_y\,\Delta y + \left(H_z + \frac{\partial H_z}{\partial y}\,\Delta y\right)\Delta z$$
$$-\left(H_y + \frac{\partial H_y}{\partial z}\,\Delta z\right)\Delta y \qquad (3{\cdot}32)$$

Carrying out the operations indicated in Equation 3·32 and simplifying, we obtain

$$\oint_l \mathbf{H} \cdot d\mathbf{l} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\Delta y\,\Delta z \qquad (3{\cdot}33)$$

However, from Equation 3·29, the line integral of **H** about a surface is equal to the current flowing through the surface. In Figure 3·2 is shown the average value of the normal component of the current density, i_x, flowing normal to the surface $\Delta y\ \Delta z$. The total current, ΔI_x, flowing through the surface is equal to the average value of the normal component of the current density times the area of the surface. Thus

$$\Delta I_x = i_x\,\Delta y\,\Delta z \qquad (3{\cdot}34)$$

This current can be equated to the line integral around the surface as expressed in Equation 3·32:

$$\oint_l \mathbf{H} \cdot d\mathbf{l} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\Delta y\,\Delta z = i_x\,\Delta y\,\Delta z \qquad (3{\cdot}35)$$

All three terms of Equation 3·35 can now be divided by the area of the surface $\Delta y\ \Delta z$:

$$\frac{\oint_l \mathbf{H} \cdot d\mathbf{l}}{\Delta y\,\Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = i_x \qquad (3{\cdot}36)$$

The values of H_y, H_z, and i_x are all average values. However, the component of the curl of a vector normal to a surface is defined as the limit, as the surface approaches zero, of the closed line integral around the surface divided by the surface; or, stated in equation form,

$$\left|\text{curl } H\right|_{\substack{\text{component}\\ \text{normal to } \Delta s}} = \operatorname*{limit}_{\Delta s \to 0}\left(\frac{\oint_l \mathbf{H} \cdot d\mathbf{l}}{\Delta s}\right) \qquad (3{\cdot}37)$$

where Δs is the surface, in this case $\Delta y\ \Delta z$. Since the surface is in the

x plane, the component of the curl obtained is the x component. It is noted as $\text{curl}_x\,\mathbf{H}$.

When the surface is allowed to approach zero, the average values become the values of the quantities at the point about which the surface vanishes. Allowing the surface $\Delta y\;\Delta z$ to approach zero, we obtain the x component of curl $\mathbf{H}$:

$$\text{curl}_x\,\mathbf{H} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = i_x \tag{3·38}$$

where H_y and H_z are the y and z components of $\mathbf{H}$ at the point in question and i_x is the x component of the current density at the same point. Hence the x component of the curl of $\mathbf{H}$ is equal to the rate of change of the z component of $\mathbf{H}$ with respect to y minus the rate of change of the y component of $\mathbf{H}$ with respect to z, and is also equal to the x component of the current density at the point in question. The y and z components of the curl of $\mathbf{H}$ can be obtained in a similar manner by taking the line integral around surfaces at right angles to those components. The values obtained are

$$\begin{aligned} \text{curl}_y\,\mathbf{H} &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = i_y \\ \text{curl}_z\,\mathbf{H} &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i_z \end{aligned} \tag{3·39}$$

where $\text{curl}_y\,\mathbf{H}$ is the y component of the curl and $\text{curl}_z\,\mathbf{H}$ is the z component of the curl. H_x is the x component of the magnetic intensity vector, $\mathbf{H}$; and i_y and i_z are the y and z components, respectively, of the current density, $\mathbf{I}$. The curl of $\mathbf{H}$ is now defined by Equations 3·38 and 3·39.

Since the curl of a vector has direction and magnitude, it is also a vector and, like any other vector, it is equal to the sum of its components, each component being multiplied by the proper unit vector. Summing up the three components, we obtain

$$\text{curl}\,\mathbf{H} = \mathbf{i}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \tag{3·40}$$

The curl, from Equations 3·38 and 3·39, is also equal to the sum of the three components of the current density multiplied by their proper unit vectors. This is equal to the current density vector, $\mathbf{I}$, so that the curl of $\mathbf{H}$ is, therefore, equal to the current density at the point in question. Thus

$$\text{curl}\,\mathbf{H} = \mathbf{I} \tag{3·41}$$

Using the definition of the vector operator nabla, ∇, given in Equation 3·21, we can express the curl of a vector as the cross product of nabla and the vector. Applying this operator for the curl of **H**, we obtain

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} \tag{3·42}$$

The cross product of nabla and **H** can be expressed by means of a determinant as demonstrated in Chapter 2, Equation 2·14. The components of the operator nabla are used in the second row taking the place of the components of the first vector of the cross product:

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \tag{3·43}$$

When the determinant operation is carried out the curl, as given by Equation 3·40, is obtained.

To obtain the curl of **E** we use Faraday's law, which states that the line integral of **E** around a closed path is equal to the negative rate of change of the magnetic flux through any surface bounded by the path. Expressing the law as an equation, we obtain

$$\oint_l \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi}{dt} \tag{3·44}$$

where ϕ is the total flux through an area bounded by the closed-path line l. Since the value of this line integral is unique, the rate of change of the total flux through any area bounded by the line will be the same. The total flux, ϕ, through any area is equal to the average value of the normal component of the magnetic induction times the area. Thus

$$\phi = B_n \Delta s \tag{3·45}$$

where Δs is the surface enclosed by the closed-path line, l, and B_n is the normal component of **B**. When the surface is allowed to approach zero, the average value as in the previous cases becomes the specific value at the point. The similarity between Equations 3·44 and 3·29 is evident so that the result may be stated immediately: The curl of **E** is equal to minus the time rate of change of the magnetic induction vector, **B**. Using the operator nabla, we obtain

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3·46}$$

This equation can be derived in the same manner as the curl of **H** equation.

In a vector field, the curl of that vector, if it does not vanish, is another vector field related to the original field by $\nabla \times$. This is true in both the cases of **H** and **E** as well as any other case concerning a vector field.

3·5 Displacement current

Four equations have been obtained so far, two divergence equations and two curl equations. Gathering these equations together, we obtain

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{I} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \tag{3·47}$$

However, the equation for the curl of **H** is not complete. Maxwell noticed this and made one of his outstanding contributions to electrical theory, his conception of displacement current. The conception of a displacement current was first needed to explain the so-called flow of alternating current through a dielectric. The reason for calling it a displacement current is the pictorial explanation first given. The dielectric was considered to be made up of positive and negative charges bound together elastically in the molecules of the material. Now, when a voltage is impressed across the faces of the dielectric, a current will flow through the dielectric. In direct current it may be the charging current of a condenser. This current was explained by stating that the charges were slightly displaced in opposite directions. However, even in a vacuum dielectric where there are no charges present the phenomenon of displacement current still exists. Maxwell pointed out that the changing electric field within the dielectric is what provides the so-called current flowing through the dielectric. Even though the explanatory idea of charges being displaced is no longer accepted, the name displacement current has remained, signifiying the equivalent current of a changing electric field.

To obtain the equation for displacement current, let us consider the condenser of capacitance C, shown in Figure 3·3. At any instant, the value of the charge, q, on the plates of the condenser is given by

$$q = CV \tag{3·48}$$

where V is the voltage across the condenser. The capacitance of the

condenser is directly proportional to the dielectric constant of the dielectric material within the condenser. Let us assume that the condenser is constructed of two parallel flat plates of area A, separated by a distance d. The intervening space is filled with a dielectric having a dielectric constant of $k_e\epsilon_0$. The capacitance of such a condenser is given by

$$C = k_e\epsilon_0 \frac{A}{d} \tag{3·49}$$

The total current, I_T, flowing through the condenser is equal to the time rate of change of charge on the plates.

$$I_T = \frac{dq}{dt} \tag{3·50}$$

Substituting Equation 3·48 into Equation 3·50, we obtain

$$I_T = C \frac{dV}{dt} \tag{3·51}$$

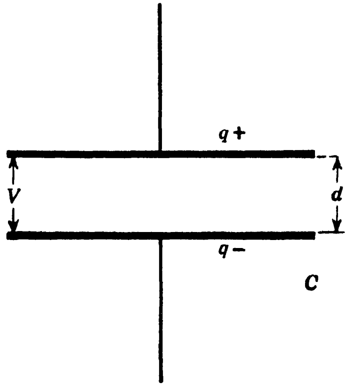


FIG. 3·3 A flat plate condenser of capacitance C for illustrating displacement current.

Equation 3·51 now has to be reduced to terms of current density and electric field intensity to fit in with the equations of Equation 3·47. The displacement current density, $\mathbf{I}_C$, is a vector equal to—assuming uniform distribution in the case of this condenser—the total current divided by the area, A:

$$\left| \mathbf{I}_C \right| = \frac{I_T}{A} \tag{3·52}$$

Similarly, if the lines of electric flux are assumed straight and evenly distributed, the voltage across the condenser is equal to the dot product of the electric field intensity and the vector distance, $\mathbf{d}$:

$$V = \mathbf{E} \cdot \mathbf{d} \tag{3·53}$$

Substituting Equations 3·49, 3·52, and 3·53 into Equation 3·51 and noting that the vector distance, $\mathbf{d}$, and the vector area, $\mathbf{A}$, have the same magnitude as A and d, we obtain for the displacement current density

$$\mathbf{I}_C = k_e\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{3·54}$$

where the partial derivative is used because **E** may be a function of other variables. But the electric induction, **D**, is equal to the product of the dielectric constant and the electric field intensity, **E**. Equation 3·54 becomes

$$\mathbf{I}_C = \frac{\partial \mathbf{D}}{\partial t} \tag{3.55}$$

Equation 3·55 demonstrates that the current density which appears to be flowing through the dielectric is equal to the rate of change of the electric flux density. When all types of currents have to be considered Equation 3·55 must be taken into account. The general expression for current density in any medium may then be written

$$\text{current density} = \mathbf{I} + \frac{\partial \mathbf{D}}{\partial t} \tag{3·56}$$

where **I** represents the conduction current density only. To complete the third equation of Equations 3·47, the displacement current is added to the conduction current present in the equation. The equation then becomes

$$\nabla \times \mathbf{H} = \mathbf{I} + \frac{\partial \mathbf{D}}{\partial t} \tag{3·57}$$

If free magnetic poles are proved to exist, the fourth equation of Equations 3·47 would have to be modified to include a magnetic conduction current in addition to the partial of **B** with respect to time. However, since it is of no practical use at the present time, this aspect is not considered.

3·6 Maxwell's equations in differential form

The vector form of notation is very useful for writing and handling the field equations of Maxwell, Equations 3·47 and 3·57. However, for rectangular coordinate calculations, it is usually easier to handle the equations when written in the complete differential form. No vector symbols are necessary in this case since the subscripts identify the separate components of the vector quantities:

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho \tag{3·58}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{3·59}$$

$$\begin{aligned}\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= i_x + \frac{\partial D_x}{\partial t}\\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= i_y + \frac{\partial D_y}{\partial t}\\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= i_z + \frac{\partial D_z}{\partial t}\end{aligned} \tag{3·60}$$

$$\begin{aligned}\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t}\\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\partial B_y}{\partial t}\\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t}\end{aligned} \tag{3·61}$$

Equation 3·58 is the divergence equation for the electric induction expressed in terms of the components of **D** at the point under consideration, and ρ is the charge density at that point. The divergence equation for the magnetic induction is given in Equation 3·59 in terms of the components of **B** at the point. This equation is equated to zero inasmuch as no free magnetic poles are considered. The curl of **H** results in three equations, Equations 3·60. These equations are obtained by using the theorem that for two vectors to be equal, their components must be equal. Equating each of the three components results in the foregoing equations. Again the values shown are the values for the point at which the curl is being taken. Repeating the same procedure for the curl of **E** equation results in the three equations of Equations 3·61. Similarly, the components shown are the components of the vectors at the point at which the curl is being taken.

However, the foregoing equations use both **E** and **D**, and both **H** and **B**. It is sometimes desirable to eliminate the induction variables, **D** and **B**. This can be accomplished by using the substitution

$$\begin{aligned}\mathbf{B} &= k_m\mu_0\mathbf{H}\\ \mathbf{D} &= k_e\epsilon_0\mathbf{E}\end{aligned} \tag{3·62}$$

Substituting the equalities of Equations 3·62 into Equations 3·58 to 3·61, inclusive, the Maxwell's equations become

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{k_e\epsilon_0} \tag{3·63}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{3·64}$$

$$
\begin{aligned}
\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= i_x + k_e\epsilon_0 \frac{\partial E_x}{\partial t} \\
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= i_y + k_e\epsilon_0 \frac{\partial E_y}{\partial t} \\
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= i_z + k_e\epsilon_0 \frac{\partial E_z}{\partial t}
\end{aligned}
\tag{3·65}
$$

$$
\begin{aligned}
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -k_m\mu_0 \frac{\partial H_x}{\partial t} \\
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -k_m\mu_0 \frac{\partial H_y}{\partial t} \\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -k_m\mu_0 \frac{\partial H_z}{\partial t}
\end{aligned}
\tag{3·66}
$$

These equations are, of course, restricted to a homogeneous, isotropic medium wherein the parameters k_e and k_m have the same value in all directions. If the medium is of the crystalline or anisotropic type, this is not true and more intricate equations have to be employed.

3·7 Maxwell's equations in alternating current form

Since, primarily, Maxwell's equations are applied to fields which vary harmonically with time, the derivative with respect to the time, t, can be carried out in the equations. As in the case of the transmission lines, it is possible to designate the harmonically varying vectors as the real part of an exponential function. As before, the symbol $\mathcal{R}$ designating the real part of the function will be implied and henceforth omitted. Thus the vector $\mathbf{E}$, if it is varying harmonically with time, may be noted as a vector $\mathbf{E}^p$ multiplied by a time factor $e^{j\omega t}$ as follows:

$$\mathbf{E} = \mathbf{E}^p e^{j\omega t} \tag{3·67}$$

The three equalities possible from the vector equation are obtained by equating similar components from each side of the equality sign:

$$
\begin{aligned}
E_x &= E_x{}^p e^{j\omega t} \\
E_y &= E_y{}^p e^{j\omega t} \\
E_z &= E_z{}^p e^{j\omega t}
\end{aligned}
\tag{3·68}
$$

where the p superscript components represent magnitudes only.

The same process may be repeated for the **H** vector:

$$\mathbf{H} = \mathbf{H}^p e^{j\omega t} \tag{3·69}$$

where $\mathbf{H}^p$ is the peak vector value of **H**. Again three equations are obtained by equating components:

$$\begin{aligned} H_x &= H_x{}^p e^{j\omega t} \\ H_y &= H_y{}^p e^{j\omega t} \\ H_z &= H_z{}^p e^{j\omega t} \end{aligned} \tag{3·70}$$

The p superscript components of **H** represent only magnitudes as in the case of the electric intensity equations. Equations 3·68 and 3·70 for **E** and **H** are true when both **E** and **H** are in phase. If they are not in phase, a modifying phase angle is added to one of them. If the **E** vector is used as the reference vector and the **H** vector lags by an angle θ, the expression for the time factor for **H** will have to incorporate the angle θ. The equation for **H** then becomes

$$\mathbf{H} = \mathbf{H}^p e^{j(\omega t - \theta)} \tag{3·71}$$

(A word of caution: The term j signifying the imaginary quantity, the square root of minus one, should not be confused with the vector quantity $\mathbf{j}$ denoting the unit vector in the positive y direction.)

As there are no time derivatives in the divergence equations, they do not change in the alternating current form. Inasmuch as the derivative of a vector is equal to the sum of the derivatives of its components, each direction being maintained by keeping the proper unit vector, the form of the time derivative may be obtained by differentiating one of the components of Equations 3·68 or 3·70. Choosing E_x and taking its derivative with respect to time, we obtain

$$\frac{\partial E_x}{\partial t} = \frac{\partial (E_x{}^p e^{j\omega t})}{\partial t} \tag{3·72}$$

However, since $E_x{}^p$ does not vary with time, the derivative of E_x is equal to $E_x{}^p$ times the derivative of $e^{j\omega t}$, which yields

$$\frac{\partial E_x}{\partial t} = j\omega (E_x{}^p e^{j\omega t}) \tag{3·73}$$

But the expression within the brackets is equal to E_x; thus

$$\frac{\partial E_x}{\partial t} = j\omega E_x \tag{3·74}$$

showing that the time derivative of the component of a vector varying harmonically with time is equal to $j\omega$ times the component. Since all the components of the two vectors are similar with respect to time, the

derivative of a vector which varies harmonically with time is equal to $j\omega$ times the vector. As before, ω is equal to 2π times the frequency. From alternating current theory it can be understood that the derivative is the same even if there were a phase angle involved. Thus, in the case of H_x at a phase angle θ,

$$H_x{}^p e^{j(\omega t-\theta)} = H_x{}^p e^{-j\theta} e^{j\omega t} \tag{3·75}$$

Hence the phase angle shift is a multiplying constant and does not change the differentiation result.

Applying this result for the time derivative to the first part of Equations 3·65, the curl of **H**, we obtain

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = i_x + j\omega k_e \epsilon_0 E_x \tag{3·76}$$

This equation can be further simplified by substituting for the conduction current, i_x, from the expression, in accordance with Ohm's law, which states that

$$i_x = \sigma E_x \tag{3·77}$$

where σ is the conductivity of the medium. Substituting into Equation 3·76, we obtain

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = (\sigma + j\omega k_e \epsilon_0) E_x \tag{3·78}$$

which is the final alternating current equation for the first part of Equations 3·65.

The form of the other two parts of Equations 3·65 is the same as the first so that they also may be written in the form of Equation 3·76. The vector equation for the curl of **H** thus becomes

$$\nabla \times \mathbf{H} = (\sigma + j\omega k_e \epsilon_0)\mathbf{E} \tag{3·79}$$

The same procedure may be repeated for the curl of **E** equations, Equations 3·66. Substituting the alternating current form for **H** and assuming no phase angle, we find that the first part of Equations 3·66 becomes

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega k_m \mu_0 H_x \tag{3·80}$$

Again, all three equations have the same form so that the vector expression for the curl of **E** is

$$\nabla \times \mathbf{E} = -j\omega k_m \mu_0 \mathbf{H} \tag{3·81}$$

Gathering together the alternating current equations developed above

and adding the divergence equations, we find that the equation of Maxwell in the alternating current vector form are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{k_e \epsilon_0} \tag{3·82}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{3·83}$$

$$\nabla \times \mathbf{E} = -j\omega k_m \mu_0 \mathbf{H} \tag{3·84}$$

$$\nabla \times \mathbf{H} = (\sigma + j\omega k_e \epsilon_0)\mathbf{E} \tag{3·85}$$

These equations may be written in the differential form by equating similar components from each side of the equal sign. Thus

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{k_e \epsilon_0} \tag{3·86}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{3·87}$$

$$\begin{aligned}
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -j\omega k_m \mu_0 H_x \\
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -j\omega k_m \mu_0 H_y \\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -j\omega k_m \mu_0 H_z
\end{aligned} \tag{3·88}$$

$$\begin{aligned}
\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= (\sigma + j\omega k_e \epsilon_0) E_x \\
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= (\sigma + j\omega k_e \epsilon_0) E_y \\
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= (\sigma + j\omega k_e \epsilon_0) E_z
\end{aligned} \tag{3·89}$$

3·8 Discussion of Maxwell's equations

Maxwell's equations, as given in Equations 3·58 to 3·61, inclusive, and the various forms derived therefrom are point relationships, inasmuch as they determine the electric and magnetic field at any point where the boundary conditions are known. They indicate all the forces that must be in balance at any point under any of the ordinary conditions of electromagnetism. All known experimental evidence, so far, indicates that all ordinary electromagnetic phenomena are governed by Maxwell's equations.

Of course the real test is the application of boundary conditions to the equations and the correspondence of the resultant solution with observed results. No single result would prove it. Only many consistently proved results have justified their application. Fundamentally, then, the solution of electromagnetic problems is taken as the solution of Maxwell's equations when applied to the problems.

Thus any electromagnetic field is completely defined by the values and distributions of the **E**, **H**, **D**, and **B** vectors. These vectors are assumed to be finite throughout the field and continuous functions of time and position at all ordinary points. Their derivatives are also assumed to be continuous. The only place where discontinuities are assumed is on those surfaces where the medium changes properties.

It has been stated that Maxwell's equations show a point relationship. It was, however, assumed that a macroscopic matter is concerned, wherein the distribution of electric charge and electric current is continuous. This condition is met for the comparatively large-scale effects which are considered in even the microwave frequency range.

Let us consider now each of Maxwell's equations and attempt to obtain a physical picture of their meaning. The divergence equations are

$$\nabla \cdot \mathbf{D} = \rho \tag{3·90}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3·91}$$

Equation 3·90 indicates that when an electric field terminates, it terminates only on a charge. The equation states that there can *not* be any change in the continuity of an electric field *without* a concentration of charge. Equation 3·91 indicates (according to present experimental data) that there are negligible isolated magnetic charges. Elementary physic courses teach that magnetic lines of force are always continuous. Every north pole is accompanied by a south pole. Thus the equation ndicates that magnetic lines of force do not terminate in any of the cases considered.

The curl equations are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3·92}$$

$$\nabla \times \mathbf{H} = \mathbf{I} + \frac{\partial \mathbf{D}}{\partial t} \tag{3·93}$$

By using the definition of curl, Equation 3·92 states that the line integral—around an infinitesimal surface—of an electric field intensity, **E**,

is equal to the time rate of change of the magnetic field passing through the surface. In Figure 3·4 is shown a flat surface *s* perpendicular to a magnetic field **B**. If this magnetic field, **B**, should change with time,

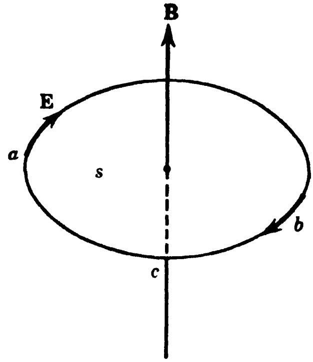


Fig. 3·4 The curl of **E** about a surface *s* perpendicular to a changing magnetic field **B**.

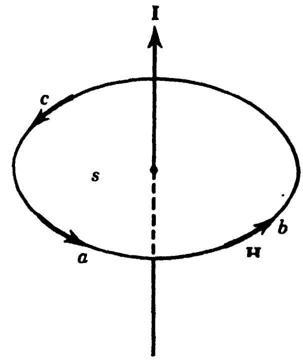


Fig. 3·5 The curl of **H** about a surface *s*, with a normal current **I** flowing through the surface.

and electric field **E** will exist in the plane of *s*, as indicated by the vectors on the line *abc*. The relationship is given by Equation 3·92. Thus, if the direction of **B** is known, the direction of **E** can be determined immediately. This **E** is present about the changing **B** whether a wire is present or not.

From well-known electrical circuit theory, a current *I*, as shown in Figure 3·5, has a magnetic field surrounding it, the magnetic lines being in a plane perpendicular to the direction of current flow. What is sometimes not realized is that a changing electric field in space is also surrounded by a magnetic field at right angles to it. This is illustrated in Figure 3·6, where **D** is changing with time. Thus, in free space, a closed loop of **H** is possible even when no conductors are present within it. It can be appreciated more fully now that a changing electric field is known to be similar in effect to an electric current as far as the magnetic field produced is concerned.

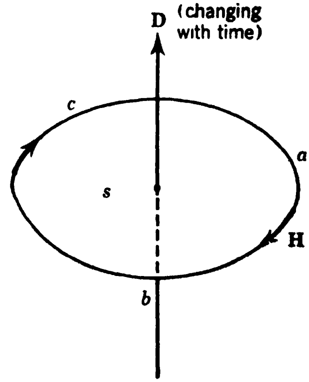


Fig. 3·6 The curl of **H** about a surface *s* cut by a changing normal electric field **D**.

3·9 The continuity of the tangential components of E and H

In Figure 3·7 is shown a small segment of the boundary between two materials, material 1 and material 2. A rectangle *abcd* is inscribed perpendicular to the surface. The size of the rectangle is l units in length and w units in width. It is so placed that it is half in material

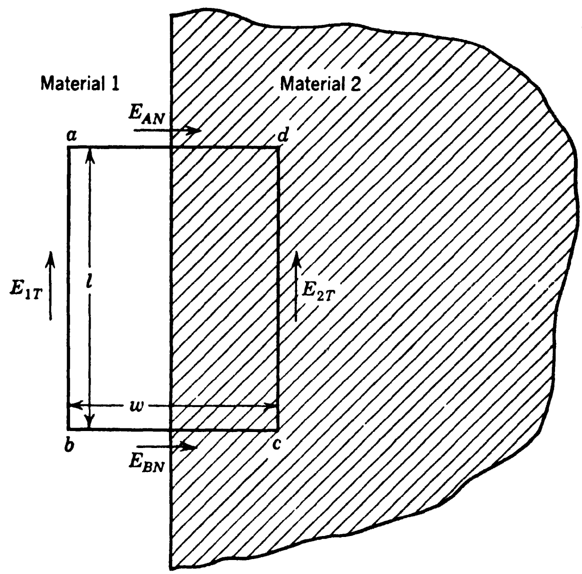


Fig. 3·7 The variation of E at the boundary of two materials, showing the normal and tangential components of the electric intensity vector.

1 and half in material 2, with the sides *ab* and *dc* parallel to the surface. E_{1T} is the average value between a and b in material 1 of the component of **E** along the line *ab*. E_{2T} is the average value between d and c in material 2 of the component of **E** along the line *dc*. Similarly E_{AN} is the average value between a and d of the component of **E** along *ad* and E_{BN} the average value between b and c of the component of **E** along *bc*. The line integral of **E** around rectangle *abcd* is equal to minus the time derivative of the surface integral of **B** over the surface of the rectangle. Taking the line integral, we obtain

$$E_{2T}l - E_{AN}w - E_{1T}l + E_{BN}w = -\frac{\partial}{\partial t}\left(\int\int_s \mathbf{B}\cdot ds\right) \qquad (3\cdot94)$$

Let us allow B_{av} to be the average value of the normal component of **B** in the rectangle. The total flux through the rectangle is then equal

to B_{av} times the area of the rectangle, l times w. This value can be substituted for the surface integral in Equation 3·94, yielding

$$E_{2T}l - E_{AN}w - E_{1T}l + E_{BN}w = -\frac{\partial B_{av}}{\partial t} lw \tag{3·95}$$

Dividing both sides of Equation 3·95 by the length l, we obtain

$$E_{2T} - E_{1T} + E_{BN}\frac{w}{l} - E_{AN}\frac{w}{l} = \frac{\partial B_{av}}{\partial t} w \tag{3·96}$$

Now let us allow the width w to approach zero in order to obtain conditions on the surface. Since l remains finite, the quantity w/l approaches zero. The only quantities which do not vanish are the average tangential components. Hence

$$E_{2T} - E_{1T} = 0 \tag{3·97}$$

or

$$E_{2T} = E_{1T} \tag{3·98}$$

The length l can now be made small enough so that the average values of the tangential components become the actual values at a point. Since w is zero, these tangential components become the tangential components at the surface. In other words, E_{1T} is the tangential component of $\mathbf{E}$ in material 1 at the surface and E_{2T} is the tangential component of $\mathbf{E}$ in material 2 at the same surface. Thus Equation 3·98 proves that the tangential component of $\mathbf{E}$ is continuous, even across surfaces of discontinuity between media.

In a similar manner, using the curl of $\mathbf{H}$ equation, we can show that

$$H_{1T} = H_{2T} \tag{3·99}$$

where H_{1T} is the tangential component of $\mathbf{H}$ in material 1 at the surface and H_{2T} is the tangential component of $\mathbf{H}$ in material 2 at the same surface point.

The divergence equations also yield two interesting results. It has been shown that when there are no charges present, the lines of electric induction are continuous. Thus, if there are no charges present on the surface of discontinuity of media,

$$D_{1N} = D_{2N} \tag{3·100}$$

where D_{1N} is the normal component of $\mathbf{D}$ in material 1 at the surface and D_{2N} is the normal component of $\mathbf{D}$ in material 2 at the same surface point. Similarly,

$$B_{1N} = B_{2N} \tag{3·101}$$

where B_{1N} is the normal component of **B** at the surface in material 1 and B_{2N} is the normal component of **B** in material 2 at the same surface point.

Care should be taken, when using Maxwell's equations, that the dielectric constant or the permeability constant should not be factored out prematurely lest erroneous results be obtained. An example is factoring out the permeability constant in the divergence of **B** equation and then applying the resultant equation to a problem concerning a discontinuity at which the permeability constant of the medium changes. The result obtained would be in error.

REFERENCE READINGS

G. W. Pierce, *Electric Oscillations and Electric Waves*, New York, McGraw-Hill Book Co., 1920, Chapters 1 and 2 of Book II.

H. H. Skilling, *Fundamentals of Electric Waves*, New York, John Wiley & Sons, 1942, Chapter IX.

S. A. Schelkunoff, *Electromagnetic Waves*, New York, D. Van Nostrand Co., 1943, Chapter 4.

R. I. Sarbacher and W. A. Edson, *Hyper and Ultrahigh Frequency Engineering*, New York, John Wiley & Sons, 1943, Chapters 1 and 2.

J. A. Stratten, *Electromagnetic Theory*, New York, McGraw-Hill Book Co., 1941, Chapters 1, 2, and 3.

S. Ramo and J. R. Whinnery, *Fields and Waves in Modern Radio*, New York, John Wiley & Sons, 1944, Chapter 4.

PROBLEMS

3·1 Derive the divergence equations in cylindrical coordinates.

3·2 Derive the divergence equations in spherical coordinates.

3·3 Derive the curl equations in cylindrical coordinates.

3·4 Derive the curl equations in spherical coordinates.

3·5 Discuss the composition of the electromagnetic field in free space a differential distance away from a perfectly conducting surface.

Chapter 4

PLANE ELECTROMAGNETIC WAVES

4·1 The electromagnetic wave

The propagation of energy by means of mechanically generated waves is a well-known concept in elementary physics. An example of this type of propagation occurs when a taut string, such as may be found in a violin or a piano, is plucked. A wave may be observed to travel down the string, be reflected, and then travel back again. Similarly, a disturbance, such as plunging a stick into a body of water, will cause waves to be generated. The water under the stick is pushed down, and as it rises and falls in an oscillatory manner some of its energy is transmitted to the adjacent water, which in turn rises and falls. This energy in turn is transmitted to the water still further out so that the rise and fall of the water, the water wave, travels outward in an ever-widening circle. We notice that the water does not travel with the wave but it is, rather, the oscillating energy contained in the rising and falling water which travels with the wave.

The concept of electromagnetic waves traveling without a guiding wire is a little more difficult to visualize. Let us first consider the interchange of energy in water waves. When the wave is at its peak the water contains an excess of potential energy. This excess causes the water to fall until it has a deficiency of potential energy at the bottom of the wave valley. At these two extreme points of excursion the water contains no kinetic energy. As the water falls from the peak to the valley, while the wave is traveling outward, the potential energy decreases and the water builds up its kinetic energy. At the instant when the water level passes the average level of the water, the wave contains no potential energy, but only kinetic energy. The same thing happens when the water is rising with the wave. It is this interchange of energy which causes the wave. The fact that the variation in motion and height of the body of water is not independent of the surrounding water causes the wave to travel outward from its source.

In electromagnetic waves a changing magnetic field will induce an electric field and, as indicated by the concept of displacement current, a changing electric field will induce a magnetic field. Also, the induced fields are not confined but will normally extend outward into the adja-

cent space. Let us imagine that a changing magnetic field exists in a region of space; hence a changing electric field also exists. The changing electric field will extend outward into the adjacent space. This electric field, in turn, will cause a changing magnetic field to be created in space still further out. The sinusoidal form of the wave causes the energy to be interchanged between the magnetic and electric fields in a manner similar to the interchange existing between the two forms of energy in the water wave. The fact that the interchange of energy is not independent of the surrounding space normally causes the wave to travel outward; unless, of course, there is a change in the medium which may reflect or absorb the wave.

There are three important vectors to be considered in the propagation of electromagnetic waves: the magnitude and direction of the **E** vector, the magnitude and direction of the **H** vector, and the magnitude and direction of the phase velocity vector. The **E** and **H** vectors are usually functions of time and position, but the velocity vector is usually a function of position only.

A plane wave is a wave which has a plane front; a cylindrical wave, one which has a cylindrical front; and a spherical wave, one which has a spherical front. The front of the wave is sometimes referred to as an equiphase surface inasmuch as it is a surface which connects points of similar phase. When a wave is generated at the antenna it is normally of the spherical type; but, at the receiving antenna, a great distance from the transmitting antenna where only a small portion of the spherical surface is intersected—a very small portion in comparison to the complete sphere—it can be treated, for all purposes, as a plane wave. This treatment is very similar to the treatment of a small portion of the circumference of a large circle as a straight line, for all purposes of calculations. In fact, the curvature is so small it can hardly be detected and, if taken into account, would not change the results perceptibly.

A ray is a line, usually straight, which is everywhere perpendicular to the equiphase surface. In a spherical wave the rays are all radial lines. In a plane wave the rays are all parallel lines.

In free space the electric and magnetic vectors are normally perpendicular to the ray. The type of wave in which this is true is known as the transverse electromagnetic wave, abbreviated *TEM* wave. Under some conditions only the electric vector is perpendicular to the ray, and then the wave is known as a transverse electric wave, abbreviated either *TE* wave or *H* wave. In other cases only the magnetic vector is perpendicular to the ray, and the wave is called a transverse magnetic wave, abbreviated either *TM* wave or *E* wave.

Maxwell's equations are used to obtain the equations for the different types of waves described. It is done usually by substituting the various boundary conditions into Maxwell's equations and solving the resultant equations. Maxwell's equations apply to all of the cases considered and they alone determine whether or not the wave exists.

4·2 The plane wave solution of Maxwell's equation

Consider the boundary conditions of a plane, *TEM* wave in free space. This, first of all, means that both the electric and the magnetic vectors are perpendicular to the ray. The coordinate system is ad-

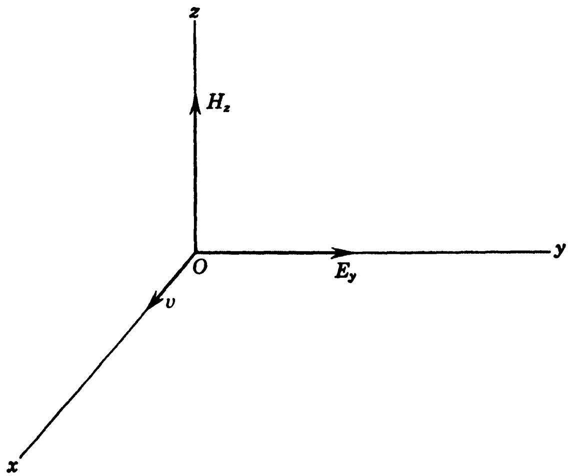


Fig. 4·1 The vectors of a plane electromagnetic wave in free space with the coordinate system aligned so that the x plane is an equiamplitude and equiphase plane.

justed so that the x plane becomes the equiphase plane; the plane determined by x equal to a constant now intersects the wave in such a manner that all points of the wave in that plane are at the same phase. In this plane wave it is also assumed that the x plane is an equiamplitude plane; hence all points at which the wave intersects the plane have the same amplitude as well as the same phase. There are two vectors involved, **E** and **H**, so that the coordinate system may also be adjusted for the **E** vector to point in the direction of y only. In other words **E** will have no x or z components. The final coordinate system is shown in Figure 4·1. In this figure **H** is drawn in the direction of $+z$. This cannot be rigorously assumed but the results will show it to be true.

In free space in Equations 3·63, 3·64, 3·65, and 3·66—the differential

forms of Maxwell's equations in **E** and **H**—it is known that

$$\begin{aligned} \rho &= 0 \\ \mathbf{I} &= 0 = i_x = i_y = i_z \\ k_e &= 1 \\ k_m &= 1 \end{aligned} \tag{4·1}$$

The charge density, ρ, is assumed zero because in free space there is no means for the charge to accumulate. Since free space is a perfect insulater, the current density, **I**, is zero. By their very definition the dielectric constant k_e and the permeability constant k_m are equal to 1.

From the way the coordinate system of Figure 4·1 is aligned, the derivatives of the **E** and **H** vectors with respec to y and z are zero. This results from the x plane being an equiphase and equiamplitude plane wherein the vectors do not vary with a variation in y or z. Substituting this result and the qualities of Equations 4·1 into Equations 3·65, the differential equations for the $\nabla \times \mathbf{H}$, we obtain

$$\begin{aligned} 0 &= \epsilon_0 \frac{\partial E_x}{\partial t} \\ -\frac{\partial H_z}{\partial x} &= \epsilon_0 \frac{\partial E_y}{\partial t} \\ \frac{\partial H_y}{\partial x} &= \epsilon_0 \frac{\partial E_z}{\partial t} \end{aligned} \tag{4·2}$$

All other terms in Equations 3·65 drop out. The first equation of Equations 4·2 merely shows, since ϵ_0 is not zero, that the partial derivative of E_x with respect to time, t, is zero at all times. For this to be true E_x itself has to be zero inasmuch as there can be no constant components of **E** in the traveling wave. Repeating the same substitution for Equations 3·66, the differential equations for the $\nabla \times \mathbf{E}$, we obtain

$$\begin{aligned} 0 &= \mu_0 \frac{\partial H_x}{\partial t} \\ \frac{\partial E_z}{\partial x} &= \mu_0 \frac{\partial H_y}{\partial t} \\ \frac{\partial E_y}{\partial x} &= -\mu_0 \frac{\partial H_z}{\partial t} \end{aligned} \tag{4·3}$$

wherein all other terms of Equations 3·66 are zero. Again the first equation merely indicates that the partial derivative of H_x with respect to the time, t, is equal to zero; it is equal to zero because the traveling wave can contain no constant values of **H**.

Thus there are only four equations from Equations 4·2 and 4·3 that can be used. However, the coordinates were aligned so that there is no z component of the vector **E**. In other words, E_z is zero for all time. The partial derivatives of zero is also zero; hence the partial derivative of E_z with respect to the time, t, and the partial derivative of E_z with respect to the coordinate x are also zero. Thus from two of the equations of Equations 4·2 and 4·3,

$$\frac{\partial H_y}{\partial t} = \frac{\partial H_y}{\partial x} = 0 \tag{4·4}$$

revealing that H_y will not vary with t or x.

It has already been shown that the **H** vector will not vary with y or with z; hence H_y will not vary with y or z. However, as stated before, the traveling wave cannot contain any fixed components of **H**. Thus H_y is identically equal to zero. Inasmuch as H_x is also identically zero, this proves the previous statement that **H** will possess only a z component and justifies drawing the **H** vector in the $+z$ direction in Figure 4·1. Now only two equations from Equations 4·3 and 4·2 are left to be considered. They are

$$\begin{aligned} -\frac{\partial H_z}{\partial x} &= \epsilon_0 \frac{\partial E_y}{\partial t} \\ -\frac{\partial E_y}{\partial x} &= \mu_0 \frac{\partial H_z}{\partial t} \end{aligned} \tag{4·5}$$

These equations are a great simplification from Maxwell's original equations. Their solution should yield the plane electromagnetic wave. The equations are very similar to the original transmission line continuity equations and will be solved in a similar manner.

Differentiating the first equation of Equations 4·5 with respect to x and the second equation with respect to t, we obtain

$$\begin{aligned} \frac{\partial^2 H_z}{\partial x^2} &= -\epsilon_0 \frac{\partial^2 E_y}{\partial x\,\partial t} \\ \frac{\partial^2 E_y}{\partial t\,\partial x} &= -\mu_0 \frac{\partial^2 H_z}{\partial t^2} \end{aligned} \tag{4·6}$$

However, the second partial of E_y taken first with respect to t and then with respect to x is equal to the second partial of E_y taken first with respect to x and then with respect to t. The order in which the partial derivatives are taken does not change the final value. This rule can

be used to combine the two foregoing equations:

$$\frac{\partial^2 H_z}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 H_z}{\partial t^2} \tag{4·7}$$

Similarly, taking the partial derivative of the first equation of Equations 4·5 with respect to t and the second with respect to x, we find that

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2} \tag{4·8}$$

The solutions of Equations 4·7 and 4·8 will be exponential functions of both x and t.

Let us assume now that the wave propagated is a harmonic function of time as stated in section 3·6. E_y is equated to the constant $E_y{}^p$ times the time function $e^{j\omega t}$, thus:

$$E_y = E_y{}^p e^{j\omega t} \tag{4·9}$$

Taking the second derivative of E_y with respect to t, we get

$$\frac{\partial^2 E_y}{\partial t^2} = \frac{\partial^2 (E_y{}^p e^{j\omega t})}{\partial t^2} = -\omega^2 E_y{}^p e^{j\omega t} \tag{4·10}$$

However, $E_y{}^p e^{j\omega t}$ is equal to E_y. Substituting this value for the second derivative of E_y with respect to t in Equation 4·8, we obtain

$$\frac{\partial^2 E_y}{\partial x^2} = -\mu_0 \epsilon_0 \omega^2 E_y{}^p e^{j\omega t} = -\mu_0 \epsilon_0 \omega^2 E_y \tag{4·11}$$

This equation states that for a constant frequency wave where ω is a constant the second partial with respect to x of E_y is equal to E_y times a constant.

The solution to this type of equation is an exponential function of x. Calling the multiplying constant of x, Γ, we find that the solution for E_y is given by

$$E_y = E_y{}^{p1} e^{\Gamma x} e^{j\omega t} \tag{4·12}$$

where $E_y{}^{p1}$ is a new constant which does not vary with x or t. However, since both E_x and E_z are zero, the vector $\mathbf{E}$ is equal to jE_y. Let us replace $jE_y{}^{p1}$ by the vector constant $\mathbf{E}^{p1}$. Thus the complete solution for the electric field intensity for a plane wave is

$$\mathbf{E} = jE_y = \mathbf{E}^{p1} e^{\Gamma x} e^{j\omega t} \tag{4·13}$$

Γ is a constant determined by the transmission constants of free space and $\mathbf{E}^{p1}$ is a vector constant determining the peak amplitude and the relative phase of the wave. Since $\mathbf{E}$ always points in the y direction,

it has only a y component indicating that the electric field intensity is always in the y direction but is a function of x and t.

To obtain the expression for Γ, we substitute E_y as given by Equation 4·12 into Equation 4·11:

$$\Gamma^2 E_y{}^{p1} e^{\Gamma x} e^{j\omega t} = -\omega^2 \mu_0 \epsilon_0 E_y{}^{p1} e^{\Gamma x} e^{j\omega t} \tag{4·14}$$

Dividing both sides of the equation by $E_y{}^{p1} e^{\Gamma x} e^{j\omega t}$, we get

$$\Gamma^2 = -\omega^2 \mu_0 \epsilon_0 \tag{4·15}$$

and taking the square root of both sides, we find that

$$\Gamma = j\omega\ (\pm\sqrt{\mu_0 \epsilon_0}) \tag{4·16}$$

Thus Γ comes out to be a pure imaginary, a function of the frequency inasmuch as ω is equal to $2\pi f$. Both μ_0 and ϵ_0 are constants. Γ is called the propagation constant in the manner of the nomenclature referring to traveling waves on transmission lines. In the case of transmission lines, Γ was considered to be a complex function made up of the attenuation constant α and the phase constant β:

$$\Gamma = \alpha + j\beta \tag{4·17}$$

The same equation is used for the propagation constant for electromagnetic waves. We see from Equation 4·16 that there is zero attenuation; hence α is equal to zero. This is logical inasmuch as free space was assumed to be a perfect insulator so that no energy is absorbed from the wave during its transmission through space. β, on the other hand, has two values, one a plus value and the other a minus value. Now let β assume the magnitude given by

$$\beta = +\omega\sqrt{\mu_0 \epsilon_0} \tag{4·18}$$

Thus the solution for Γ is $+j\beta$ and $-j\beta$.

Whenever there are two solutions to an exponential function, as in the case of Γ, there are two arbitrary constants involved, resulting in two solutions, the complete solution being the sum of the two individual solutions. Let one constant be $\mathbf{E}^+$ and the other $\mathbf{E}^-$ (similar to the constants used in the case of the transmission line solutions). The plus and minus superscripts are used because it will be shown that they are two traveling waves, one traveling in the plus direction and the other traveling in the minus direction. We notice that the constants are vectors. They have to be in order for their sum to be equal to a vector, in this case the electric field intensity vector, $\mathbf{E}$. The complete solution for $\mathbf{E}$, with the time variable factored out, thus becomes

$$\mathbf{E} = (\mathbf{E}^+ e^{-j\beta x} + \mathbf{E}^- e^{+j\beta x}) e^{j\omega t} \tag{4·19}$$

Since the expression for H_z given in Equation 4·7 is similar to the equation for E_y as given in Equation 4·8 (of which the foregoing is a solution), the solution for H_z can be written down immediately. Also, inasmuch as H_z is the only component of $\mathbf{H}$ which is not equal to zero, the result can be written in terms of the vectors involved as in Equation 4·19, yielding for the magnetic intensity vector

$$\mathbf{H} = (\mathbf{H}^+ e^{-j\beta x} + \mathbf{H}^- e^{+j\beta x})e^{j\omega t} \tag{4·20}$$

The phase constant, β, has the same magnitude as in the electric intensity solution, Equation 4·18. $\mathbf{H}^+$ and $\mathbf{H}^-$ are vector constants which determine the magnitude and relative phase of the two traveling magnetic intensity waves. As in the case of transmission lines, the time factor $e^{j\omega t}$ may be omitted so that the solutions become

$$\begin{aligned} \mathbf{E} &= \mathbf{E}^+ e^{-j\beta x} + \mathbf{E}^- e^{j\beta x} \\ \mathbf{H} &= \mathbf{H}^+ e^{-j\beta x} + \mathbf{H}^- e^{j\beta x} \end{aligned} \tag{4·21}$$

These are the traveling wave equations for the plane electromagnetic wave. The phase factor, $e^{-j\beta x}$, indicates a delay in phase as x is increased. If the equiphase planes with the $e^{-j\beta x}$ factor are plotted with time as a variable, they will, as time increases, move outward in the plus x direction. Similarly, the equiphase planes with the $e^{j\beta x}$ terms move in the direction of minus x when the time, t, increases. Equations 4·21 are the same type of equations that were discussed in detail in section 1·6 of Chapter 1.

When β is expressed in the number of radians per meter, the velocity, v, with which the plane wave is traveling in the direction of its ray is given by

$$v = \frac{\omega}{\beta} \text{ meters per second} \tag{4·22}$$

If the velocity should be measured along any other line making an angle of, let us say, α to the ray, a velocity larger than the true velocity will be obtained. It will be equal to the true velocity divided by the cosine of the angle α. A physical picture of this phenomenon can be obtained by placing together two pencils and a ruler, the ruler on top of the pencils. One pencil is placed so that its length is normal to the edge of the ruler, the other not normal to the edge of the ruler. As the edge of the ruler is moved at right angles to the normal, in other words along the pencil which is perpendicular to the edge, the edge will traverse the pencil not normal to it with a velocity greater than the true velocity of the ruler. The reason for this greater velocity is that there are two components concerned, the velocity with which the intersec-

tion between the pencil and the edge is moving and the velocity of the ruler itself.

The same error is encountered if the velocity of an electromagnetic wave is measured along any other line but its ray line, the line perpendicular to the equiphase plane. The actual value of the velocity can be obtained by substituting for β as given in Equation 4·18 into Equation 4·22:

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{4·23}$$

Substituting for μ_0 and ϵ_0 their values as given in Equations 3·4 and 3·7, we obtain for the velocity 3×10^8 meters per second. If the exact values of μ_0 and ϵ_0 were used, the result would be the same as the measured velocity of light in free space. This is what first indicated that light is also an electromagnetic wave. A medium, the ether, was assumed for the transmission of light waves leading to the assumption that electromagnetic waves were also transmitted in that indefinable medium, the ether, which fills all space. Actually the necessity of defining a medium for the transmission of the electromagnetic wave has become negligible because of all the definite proofs that electromagnetic waves exist and can be used irrespective of their medium of transmission. Whether the medium exists or not is unimportant and does not seem to worry scientists or engineers any more; but the fact that electromagnetic waves do exist and travel through free space at a definite rate, following the conditions laid down by Maxwell's equations, is now generally accepted.

Not all the constants of the traveling wave equations, Equations 4·21, are independent. Out of the four constants, $\mathbf{E}^+$, $\mathbf{E}^-$, $\mathbf{H}^+$, and $\mathbf{H}^-$, only two are independent. This dependency can be shown by substituting the solutions obtained in Equations 4·21 into either equation of Equations 4·5, recalling at the same time that $\mathbf{H}$ has only one component H_z and $\mathbf{E}$ has only one component E_y. Although $e^{j\omega t}$ is not shown in the equations, it is used to obtain the time derivatives of the functions. The derivatives to be used are

$$\begin{aligned} \frac{\partial H_z}{\partial x} &= \frac{\partial |\,\mathbf{H}\,|}{\partial x} = -j\beta |\,\mathbf{H}^+\,| e^{-j\beta x} + j\beta\,|\,\mathbf{H}^-\,| e^{j\beta x} \\ \frac{\partial E_y}{\partial t} &= \frac{\partial |\,\mathbf{E}\,|}{\partial t} = j\omega |\,\mathbf{E}^+\,| e^{-j\beta x} + j\omega |\,\mathbf{E}^-\,| e^{j\beta x} \end{aligned} \tag{4·24}$$

Inasmuch as the vectors have only one component, the derivatives of the components are equal to the derivatives of the magnitudes of the vectors. Putting in directions by using vectors will not affect the re-

sults provided the symbols meaning the magnitudes of the vectors are used throughout the solution of the equations. The first equation of Equations 4·5 then becomes

$$+j\beta H^+e^{-j\beta x} - j\beta H^-e^{j\beta x} = j\omega\epsilon_0 E^+e^{-j\beta x} + j\omega\epsilon_0 E^-e^{j\beta x} \qquad (4\cdot25)$$

Collecting similar exponential terms, we obtain

$$(\omega\epsilon_0 E^+ - \beta H^+)je^{-j\ x} + (\omega\epsilon_0 E^- + \beta H^-)je^{+j\beta x} = 0 \qquad (4\cdot26)$$

Equation 4·26 states that the sum of two exponential functions is equal to zero. Since this equation was derived directly from Maxwell's equations, it is true throughout the field or, more explicitly, for all values of x. However, one exponential, $e^{j\beta x}$, increases with an increase in x whereas the other, $e^{-j\beta x}$, decreases with an increase in x. The only way for their sum to be zero is for both coefficients to be zero. Consequently,

$$\begin{aligned} \omega\epsilon_0 E^+ - \beta H^+ &= 0 \\ \omega\epsilon_0 E^- + \beta H^- &= 0 \end{aligned} \qquad (4\cdot27)$$

Replacing β by its value from Equation 4·18 and solving for E^+ and E^-, we obtain

$$\begin{aligned} E^+ &= \sqrt{\frac{\mu_0}{\epsilon_0}}\, H^+ \\ E^- &= -\sqrt{\frac{\mu_0}{\epsilon_0}}\, H^- \end{aligned} \qquad (4\cdot28)$$

This indicates that the magnitude of the $\mathbf{E}^+$ vector constant is equal to $\sqrt{\mu_0/\epsilon_0}$ times the $\mathbf{H}^+$ vector constant. Similarly, the $\mathbf{H}^-$ vector constant is related in magnitude to the $\mathbf{E}^-$ vector constant by a factor $-\sqrt{\mu_0/\epsilon_0}$.

The magnetic intensity can be considered to be similar to a current inasmuch as it is a manifestation of a current. Keeping this in mind and comparing Equations 4·28 to the transmission line equations (Equations 1·32) in Chapter 1 where the magnitudes of the current traveling waves and the voltage traveling waves are related by the characteristic impedance of the transmission line, we find that the form is exactly the same. For this reason, the quantity $\sqrt{\mu_0/\epsilon_0}$ is often referred to as the intrinsic impedance of free space.

From the result showing that the electric intensity vector possesses only a y component and the magnetic intensity vector possesses only a z component, it can be said that in a plane wave the electric intensity vectors are always at right angles to the magnetic intensity vectors.

When the plane wave propagation does not take place in free space, the constants k_e and k_m may not be equal to one. If the material through which it is being propagated is lossless, the derivation will be exactly the same except that everywhere ϵ_0 appears it will be multiplied by k_e and everywhere μ_0 appears it will be multiplied by k_m. Following these rules, we obtain as a solution for a lossless medium wherein k_e and k_m need not be equal to one.

$$\begin{aligned} \mathbf{E} &= \mathbf{E}^+ e^{-j\beta x} + \mathbf{E}^- e^{j\beta x} \\ \mathbf{H} &= \mathbf{H}^+ e^{-j\beta x} + \mathbf{H}^- e^{j\beta x} \end{aligned} \tag{4·29}$$

where

$$\begin{aligned} \beta &= \omega\sqrt{k_e k_m \mu_0 \epsilon_0} \\ E^+ &= \sqrt{\frac{k_m \mu_0}{k_e \epsilon_0}}\, H^+ \\ E^- &= -\sqrt{\frac{k_m \mu_0}{k_e \epsilon_0}}\, H^- \\ v &= \frac{1}{\sqrt{k_e k_m \epsilon_0 \mu_0}} \end{aligned} \tag{4·30}$$

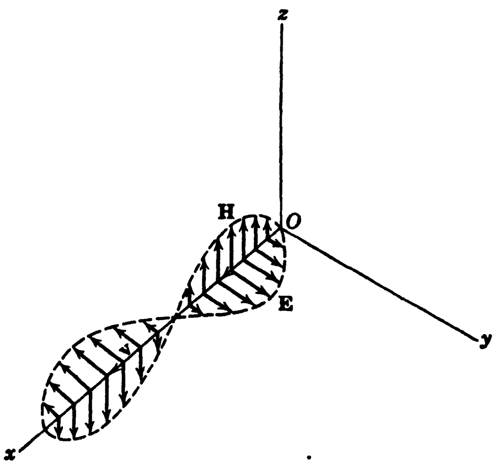


FIG. 4·2 Vector relationships in a plane electromagnetic wave at one instant of time.

Equations 4·30 shows that the intrinsic impedance, which in this case is equal to $\sqrt{k_m\mu_0/k_e\epsilon_0}$, will increase as the permeability constant increases and will decrease as the dielectric constant increases. The

velocity, on the other hand, decreases with an increase of either the permeability constant or the dielectric constant. Since the velocity decreases in the medium and the frequency remains the same, the wavelength in the medium, $2\pi/\beta$, will decrease with an increase in either of the constants.

In Figure 4·2 is presented a diagram of the vector relationships at one instant along a ray of the plane wave. Only one of the traveling waves is shown. We notice that **E**, **H**, and **v** are mutually perpendicular and that **E** and **H** go through zero at the same point, being in phase along the x axis. This is true because their magnitudes are related by a magnitude, the intrinsic impedance of space. In a lossless medium this impedance contains no phase angle.

EXAMPLE 4·1 Determine the intrinsic impedance of free space.

The intrinsic impedance is given by the square root of the permeability constant of free space divided by the dielectric constant of free space. Substituting their values from Equations 3·4 and 3·7, we obtain

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$Z_0 = \sqrt{\frac{4\pi \times 10^{-7}}{\frac{1}{36\pi} \times 10^{-9}}}$$

$$Z_0 = 377 \text{ ohms} \qquad \textit{Ans.}$$

EXAMPLE 4·2 A plane wave is transmitted through a medium which has a comparative dielectric constant of 2.3 and a comparative permeability constant of 1. Determine the intrinsic impedance of the medium and the velocity at which the wave travels through the medium.

In a dielectric material such as this, equations 4·30 have to be used. The intrinsic impedance Z_0 is given by

$$Z_0 = \sqrt{\frac{\mu_0}{k_e\epsilon_0}}$$

$$Z_0 = \sqrt{\frac{4\pi \times 10^{-7}}{2.3 \times \frac{1}{36\pi} \times 10^{-9}}}$$

$$Z_0 = 249 \text{ ohms} \qquad \textit{Ans. (a)}$$

The velocity, v, is given by

$$v = \frac{1}{\sqrt{k_e\epsilon_0\mu_0}}$$

$$v = \frac{3 \times 10^8}{\sqrt{k_e}}$$

$$v = 1.98 \times 10^8 \text{ meters per second} \qquad \textit{Ans. (b)}$$

4·3 Poynting's radiation vector

To show the convenience of the vector handling of equations, vector analysis reasoning is used throughout this section. It will be the only section where the vector analysis reasoning is employed, but it should be noted how this type of manipulation simplifies the mathematical handling of the equations.

Before deriving the radiation vector, we must prove the vector relation

$$\nabla \cdot \mathbf{E} \times \mathbf{H} = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} \tag{4·31}$$

Equation 4·31 is true if expanding both sides, in terms of the components of the vectors and their derivatives, results in identical equations. The cross product must always be worked out first in the above equation or there will be the problem of taking the cross product of a vector and scalar—an impossibility. Expanding the left-hand side of Equation 4·31, we obtain

$$\nabla \cdot \mathbf{E} \times \mathbf{H} = \frac{\partial}{\partial x}(E_yH_z - E_zH_y) + \frac{\partial}{\partial y}(E_zH_x - E_xH_z) + \frac{\partial}{\partial z}(E_xH_y - H_xE_y) \tag{4·32}$$

This can be simplified by differentiating the products and collecting terms:

$$\begin{aligned}\nabla \cdot \mathbf{E} \times \mathbf{H} = H_x\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + H_y\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) \\ + H_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) - E_x\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) \\ - E_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) - E_z\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\end{aligned} \tag{4·33}$$

Now considering the right-hand side of Equation 4·31 and expanding it, we obtain

$$\begin{aligned}\mathbf{H}\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\nabla\times\mathbf{H} &= H_x\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+H_y\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\\ &+H_z\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)-E_x\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)\\ &-E_y\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)-E_z\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\end{aligned} \tag{4·34}$$

Comparing Equation 4·34 with Equation 4·33, we find that they are identical, proving the relationship in Equation 4·31.

Returning now to the discussion of power, we shall examine the small segment of volume $\Delta\tau$ shown in Figure 4.3, where

$$\Delta\tau = \Delta x\,\Delta y\,\Delta z \tag{4·35}$$

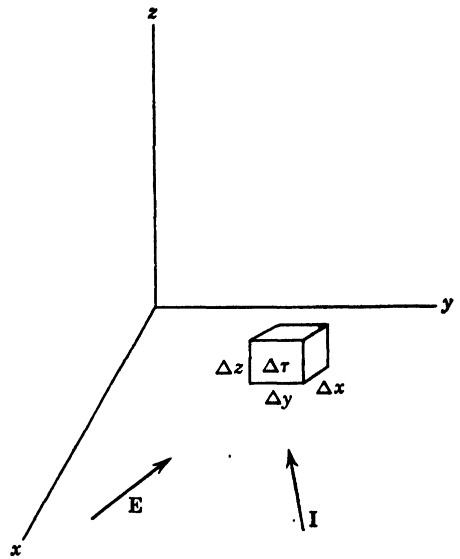


FIG 4·3 A small segment of volume $\Delta\tau$ in a field of electric field intensity **E** and current intensity **I**.

Let us call **E** the average electric field intensity throughout the volume, $\Delta\tau$, its components being the average value of the true components of the actual field intensity; **I** the average value of the current density in the volume, $\Delta\tau$, its components being the average values of the components of the actual current density in the volume. The power, P, being dissipated, can be calculated for any direction; it is the product of the electromotive force in the desired direction and the conduction current flowing in that direction.

The y component of V, ΔV_y, is equal to the y component of the electric field intensity, E_y, times the thickness, in the y direction, of the volume $\Delta\tau$. This results in

$$\Delta V_y = E_y \, \Delta y \tag{4·36}$$

The total conduction current flowing in the y direction, ΔI_y, is equal to the y component of the conduction current density times the area through which it flows. This current density, i_y, is taken as the conduction current only which is flowing.

$$\Delta I_y = i_y \Delta x \, \Delta z \tag{4·37}$$

The total power being dissipated in the volume by the conduction current flowing in the y direction is equal to the product of Equations 4·37 and 4·36. Hence the power dissipated, ΔP_y, is given by

$$\Delta P_y = E_y i_y \, \Delta x \, \Delta y \, \Delta z \tag{4·38}$$

In a plane wave oriented as shown in Figure 4·1, this would be the only component that would exist; but in the general case, the electric field intensity would possess all three components. If this should be true, the x component of the power, ΔP_x, and the z component, ΔP_z, would be obtained in a similar manner, yielding

$$\begin{aligned} \Delta P_x &= E_x i_x \, \Delta x \, \Delta y \, \Delta z \\ \Delta P_z &= E_z i_z \, \Delta x \, \Delta y \, \Delta z \end{aligned} \tag{4·39}$$

The total power, ΔP, dissipated in the volume $\Delta\tau$ is obtained by taking the sum of the three components as given in Equations 4·38 and 4·39. Taking the sum and factoring out $\Delta x \, \Delta y \, \Delta z$, which is equal to $\Delta\tau$, we obtain

$$\Delta P = (E_x i_x + E_y i_y + E_z i_z)\Delta\tau \tag{4·40}$$

Dividing both sides of the equation by $\Delta\tau$ and letting the volume $\Delta\tau$ approach zero, we find that

$$\frac{dP}{d\tau} = \mathbf{E} \cdot \mathbf{I} \tag{4·41}$$

E is now the actual electric field intensity and **I** is now the actual conduction current density at the point about which the volume $\Delta\tau$ was allowed to approach zero. This equation shows that the derivative of the power in any medium with respect to the volume is equal to the dot product of the electric field intensity and the current density.

Repeating the curl **H** equation from Maxwell's equations,

$$\nabla \times \mathbf{H} = \mathbf{I} + \frac{\partial \mathbf{D}}{\partial t} \tag{4·42}$$

and solving for the conduction current density **I**, we get

$$\mathbf{I} = \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \tag{4·43}$$

This equation for the current density—wherein the current density is equal to the difference between the curl of the magnetic intensity, **H**, and the partial with respect to the time, t, of the electric induction, **D** —is correct for any medium since Maxwell's equations apply to all mediums. Replacing the conduction current density in Equation 4·41 by its equivalent from Equation 4·43, we obtain for the derivative of the power with respect to volume

$$\frac{dP}{d\tau} = \mathbf{E} \cdot \nabla \times \mathbf{H} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{4·44}$$

Using now the identity noted at the beginning of the section, Equation 4·31, we obtain a symmetrical form

$$\frac{dP}{d\tau} = \mathbf{H} \cdot \nabla \times \mathbf{E} - \nabla \cdot \mathbf{E} \times \mathbf{H} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{4·45}$$

Again, from Maxwell's equations, the curl of **E** is equal to the negative of the partial of **B** with respect to time, t. Substituting this equality into Equation 4·45, we find that

$$\frac{dP}{d\tau} = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot \mathbf{E} \times \mathbf{H} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{4·46}$$

However **B** is equal to $k_m \mu_0 \mathbf{H}$. Simplifying the first term on the right-hand side of Equation 4·46, we get

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{H} \cdot k_m \mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{4·47}$$

From vector considerations, since the result is the same, this can now be written as the derivative of the square of the magnitude, H, of the vector **H**:

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}\left(k_m \mu_0 \frac{H^2}{2}\right) \tag{4·48}$$

Similarly the third term of Equation 4·46 can be simplified to the derivative of the square of the magnitude, E, of the vector **E**:

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{k_e \epsilon_0 E^2}{2}\right) \tag{4·49}$$

Equations 4·48 and 4·49 can now be substituted into Equation 4·46 to obtain the power as the sum of the derivatives of two scalar quantities and the divergence of a cross product. Combining terms, factoring out the time derivative, and multiplying both sides of the equation by the differential volume, $d\tau$, we obtain the equation for the power in the differential volume:

$$dP = -\frac{\partial}{\partial t}\left(\frac{k_m\mu_0 H^2}{2} + \frac{k_e\epsilon_0 E^2}{2}\right) d\tau - (\nabla \cdot \mathbf{E} \times \mathbf{H}) d\tau \tag{4·50}$$

However, the second term of the right-hand side of the equation is the divergence of a vector. The detail that this vector happens to be the cross product of two other vectors does not affect the result when the surface integral over the surface, s, of the volume involved is substituted for the divergence. This is true for the divergence of any vector as stated in the definition of divergence. Replacing the divergence in Equation 4·50 by the equivalent surface integral, we find that the power equation resolves into

$$dP = -\frac{\partial}{\partial t}\left(\frac{k_m\mu_0 H^2}{2} + \frac{k_e\epsilon_0 E^2}{2}\right) d\tau - \oint\!\!\oint_s (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} \tag{4·51}$$

The power can now be considered to be made up of two terms. The first is the time derivative of a quantity which will be called P_v, where

$$P_v = \frac{k_m\mu_0 H^2}{2} + \frac{k_e\epsilon_0 E^2}{2} \tag{4·52}$$

This term is a volume term and represents the power contained in the static fields of **E** and **H**. There is a minus sign in front of the term in the equation for power. The change in this term with respect to time multiplied by the differential volume may be taken as representing the decrease in the amount of magnetic and electrical energy stored in the differential volume $d\tau$.

The portion inside the integral of second term for power in Equation 4·51 will be called $\mathbf{P}_r$, where

$$\mathbf{P}_r = \mathbf{E} \times \mathbf{H} \tag{4·53}$$

The power in this term, the negative of the surface integral as applied in Equation 4·51, represents the energy moving inward through the surface of the volume. Notice that $\mathbf{P}_r$ is a vector having both magnitude and direction. It is usually called Poynting's radiation vector after its discoverer, Professor J. H. Poynting. Interpreting the vector terminology we see that it represents a vector at right angles to both **E** and **H** and pointing in the direction a right-handed screw would

move if **E** were turned into **H** through the smallest angle between them. This vector defines the magnitude and the direction of the flow of power in an electromagnetic wave.

In free space and in other perfect insulators the actual conduction current flowing is zero. Thus the power absorbed by the medium is zero; dP is zero. In this case

$$\frac{\partial}{\partial t}(P_v) = -\oint\oint_s \mathbf{P}_r \cdot d\mathbf{s} \tag{4.54}$$

stating that the change in the electrical and magnetic energy stored in the medium is equal to the electromagnetic energy flowing into the volume through the surface. In this manner electric and magnetic fields are built up around a receiving antenna by means of electromagnetic energy flowing through space.

EXAMPLE 4·3 Determine the value of Poynting's radiation vector in a traveling electromagnetic wave having an electric field intensity of one volt per meter. The medium is free space.

In a traveling electromagnetic wave the electric vector and the magnetic vector are at right angles. Hence the magnitude of the cross product of **E** and **H** will be equal to the product of the magnitudes of the vectors. Thus

$$|\mathbf{P}_r| = |\mathbf{E} \times \mathbf{H}| = EH$$

But from Equation 4·28 the magnitude of **H** is related to the magnitude of **E** by the intrinsic impedance of free space. Substituting for H its equivalent, E divided by the intrinsic impedance of free space (377 ohms), we obtain

$$\mathbf{P}_r = E\frac{E}{377} = 1\frac{1}{377} = 2.65 \times 10^{-3} \text{ watts per square meter} \qquad \textit{Ans.}$$

4·4 REFLECTION FROM A PLANE OF ARBITRARY INCIDENCE

Figure 4·4 illustrates a plane wave whose ray W_1O makes an angle θ_1 with the normal of a perfectly conducting plane surface M. The reflected wave is assumed to have a ray OW_3 that makes an angle θ_3 to the normal of the plane M. Any point on the incident wave must satisfy the condition

$$lx + my + nz = s \tag{4.55}$$

where l, m, and n are the direction cosines of a ray. Since all rays in a plane wave are parallel, it will apply to all rays in the plane wave. The distance, s, is measured from the point at which the ray intersects the plane, M, to the equiphase wave front under consideration. Also, inasmuch as the wave under discussion consists of a wave traveling in only one direction, it can be expressed as a function of the time, t, minus

the distance, s, divided by the velocity, v, of the wave. Calling the vector function $\mathbf{F}_1$ and using it to define the electric intensity vector, $\mathbf{E}_1$, we may write

$$\mathbf{E}_1 = \mathbf{F}_1\left(t - \frac{s}{v}\right) \tag{4·56}$$

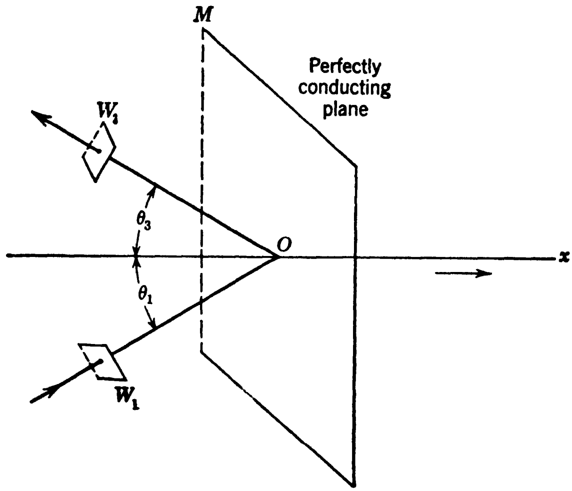


FIG. 4·4 Reflection of a plane electromagnetic wave W from a perfectly conducting plane surface.

The components of Equation 4·56 may be written using the subscripts $1x$, $1y$, and $1z$. For Equation 4·56 to be true, all values of similar components on opposite sides of the equation must be equal. Thus

$$\begin{aligned} E_{1x} &= F_{1x}\left(t - \frac{s}{v}\right) \\ E_{1y} &= F_{1y}\left(t - \frac{s}{v}\right) \\ E_{1z} &= F_{1z}\left(t - \frac{s}{v}\right) \end{aligned} \tag{4·57}$$

where F_{1x}, F_{1y}, and F_{1z} are the functions of $\left(t - \frac{s}{v}\right)$, components of the vector function $\mathbf{F}_1$.

It has been shown before, in section 3·8 of Chapter 3, that the tangential components of the electric intensity vector, $\mathbf{E}$, must be the same on both sides of a surface. However, inside a perfectly conducting

surface the electric intensity vector must be everywhere zero inasmuch as an electric current would immediately flow to equalize it.

Let us assume, as shown in Figure 4·4, that the normal to the surface is taken as the x direction in the system of coordinates being used. The coordinate system can always be arranged to make this true. Thus when x is equal to zero, the total tangential electric intensity must be equal to zero. The total electric intensity at the surface (x equal to zero) is made up of the incident wave impinging on the surface plus the reflected wave from the surface. Only if the reflected wave has a magnitude and phase which will just neutralize the tangential component of the incident wave will the total electric intensity on the surface be zero. Inasmuch as the normal to the surface was chosen as the x direction, the y and z components are the tangential components.

The reflected wave can also be expressed as a function of the velocity of the wave, v, the time, t, and the distance, s'. The distance s' is the distance from the point of intersection of the reflected-wave ray OW_3 and the plane M to the equiphase wave front under consideration, measured along the ray OW_3. This definition assumes that the reflected wave is also a plane wave, a logical assumption. Using the subscript 3 for identification, we may write for the reflected wave and its components

$$\begin{aligned} \mathbf{E}_3 &= \mathbf{F}_3\left(t - \frac{s'}{v}\right) \\ E_{3x} &= F_{3x}\left(t - \frac{s'}{v}\right) \\ E_{3y} &= F_{3y}\left(t - \frac{s'}{v}\right) \\ E_{3z} &= F_{3z}\left(t - \frac{s'}{v}\right) \end{aligned} \tag{4·58}$$

Returning now to the discussion of the tangential components of the electrical field intensity on the surface of the plane, we find that the sum of the y components of the incident and the reflected waves and the sum of the z components of the incident and the reflected waves must equal zero. Taking the sum of the y and z components of Equations 4·57 and 4·58, and equating them to zero, we obtain

$$\begin{aligned} 0 &= F_{1y}\left(t - \frac{s_0}{v}\right) + F_{3y}\left(t - \frac{s_0'}{v}\right) \\ 0 &= F_{1z}\left(t - \frac{s_0}{v}\right) + F_{3z}\left(t - \frac{s_0'}{v}\right) \end{aligned} \tag{4·59}$$

where s_0 and s_0' are all points at which x is equal to zero. All these are points on the plane M. Calling y_0 and z_0 any point on the plane, we obtain

$$\begin{aligned} s_0 &= m_1 y_0 + n_1 z_0 \\ s_0' &= m_3 y_0 + n_3 z_0 \end{aligned} \tag{4·60}$$

where m_1 and n_1 are two direction cosines of the ray W_1O. Similarly, m_3 and n_3 are two of the direction cosines of the ray OW_3 along which s' is measured. Substituting Equation 4·60 into Equation 4·59, we obtain

$$\begin{aligned} 0 &= F_{1y}\left(t - \frac{m_1 y_0 + n_1 z_0}{v}\right) + F_{3y}\left(t - \frac{m_3 y_0 + n_3 z_0}{v}\right) \\ 0 &= F_{1z}\left(t - \frac{m_1 y_0 + n_1 z_0}{v}\right) + F_{3z}\left(t - \frac{m_3 y_0 + n_3 z_0}{v}\right) \end{aligned} \tag{4·61}$$

Equations 4·61 must be true for all values of t. For this to be correct, the following equalities must exist:

$$\begin{aligned} F_{3y} &= -F_{1y} \\ F_{3z} &= -F_{1z} \\ m_3 &= m_1 \\ n_3 &= n_1 \end{aligned} \tag{4·62}$$

Hence the two functions, two components of the reflected wave, are the negative of the incident wave components. The direction cosines of the y and z components are the same as the incident wave. Thus the reflected wave is in the plane of the incident wave ray and the normal to the surface M.

However, the sum of the squares of the direction cosines of any straight line is equal to one. Therefore, the sum of the squares of the direction cosines of the ray W_1O and the sum of the squares of the direction cosines of the ray OW_3 are both equal to one and they may be equated to one another, yielding

$$l_1^2 + m_1^2 + n_1^2 = l_3^2 + m_3^2 + n_3^2 \tag{4·63}$$

The subscript 3 identifies the direction cosines of the ray OW_3. But m_1 is equal to m_3 and n_1 is equal to n_3 so that Equation 4·63 reduces to

$$l_1^2 = l_3^2 \tag{4·64}$$

Taking the square root of both sides of the equation, we find that

$$l_3 = \pm\, l_1 \tag{4·65}$$

Hence there are two solutions, one wherein l_3 is equal to l_1 and the other wherein l_3 is equal to $-l_1$. If l_3 is taken equal to l_1, it means that s' is identical with s or that the ray W_1O and the ray OW_3 would coincide. However, the y and z components are equal and opposite so that the sum of these field intensity components would everywhere be zero. No wave would be reflected inasmuch as this would leave the wave with only an x component; the wave would be traveling parallel to the plane M. When l_3 is taken equal to $-l_1$, the law of reflection from a perfectly conducting surface is obtained. This law states that the reflected ray is in the plane determined by the incident ray and the normal to the surface. Furthermore, the angle of reflection is equal to the angle of incidence.

Concerning the intensity of the reflected wave, all that we have to determine is the function F_{3x}. It can be obtained by using the principle that the electric intensity in the wave front of the reflected wave is the same as in the wave front of the incident wave. Using the equalities of Equation 4·62 and the result that l_3 is equal to $-l_1$, we find that F_{3x} is equal to F_{1x}. Thus we come to the interesting conclusion that at the surface of the perfectly conducting plane the normal component of $\mathbf{E}$ is doubled while the other components are reduced to zero.

Similarly, it can be shown that the normal magnetic intensity is reduced to zero whereas the tangential magnetic intensity is doubled.

4·5 Refraction of an electromagnetic wave

If the reflecting surface is not a perfect conductor but rather a perfect dielectric with a dielectric constant and perhaps a permeability constant different from the constants of the first medium, the reflected ray will still follow the direction given by the law of reflection but the refracted ray will follow the direction determined by the index of refraction, a factor proportional to the velocities of the waves in each of the two materials. An incident ray, A, is shown in Figure 4·5 impinging on a surface, S, dividing two materials, material 1 and material 2. The electromagnetic wave velocity, v_1, in material 1 and the electromagnetic wave velocity, v_2, in material 2 are given by

$$v_1 = \frac{v_0}{\sqrt{k_{e1}k_{m1}}}$$
$$v_2 = \frac{v_0}{\sqrt{k_{e2}k_{m2}}} \tag{4·66}$$

The term v_0 is the velocity of an electromagnetic wave in free space and the subscripts 1 and 2 apply to the respective mediums involved. The

refraction angle, θ_2, is related to the angle of incidence, θ_1, by the following equation:

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2} \tag{4·67}$$

This equation is stated without proof, being a well-known relationship for light waves.

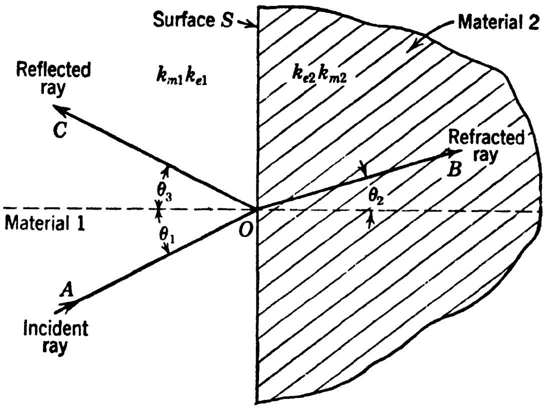


FIG. 4·5 The reflection and refraction of a plane electromagnetic wave at the surface dividing two electrically different media.

Fresnel's equations give the relationship between the magnitude of the reflected wave, E_3, and the magnitude of the incident wave, E_1, in terms of the angles θ_2 and θ_1. As shown, the magnitude of the electric intensity is used to define the magnitude of the wave inasmuch as the magnetic intensity is related to the electric intensity by the intrinsic impedance of the material. Two cases are possible, one with the electric vector parallel to the surface and the other with the magnetic vector parallel to the surface. All other cases can be considered to be combinations of these two. For an incident plane wave with the electric vector perpendicular to the plane of incidence, i.e., parallel to the surface of refraction, the electric vector magnitudes are related by

$$\frac{E_3}{E_1} = \frac{\sin (\theta_1 - \theta_2)}{\sin (\theta_1 + \theta_2)} \tag{4·68}$$

For an incident plane wave with the magnetic vector perpendicular to the plane of incidence, i.e., parallel to the surface of refraction, the electric vector magnitudes are related by

$$\frac{E_3}{E_1} = \frac{\tan (\theta_1 - \theta_2)}{\tan (\theta_1 + \theta_2)} \tag{4·69}$$

EXAMPLE 4·4 Determine the angle of reflection and the angle of refraction when a plane electromagnetic wave, with its electric vector perpendicular to the plane of incidence, impinges in free space upon the surface of a dielectric wherein k_e is equal to 4 and k_m is equal to 1. The angle of incidence is 45°. Determine also the ratio between the reflected electric intensity and the incident electric intensity.

The angle of reflection can be stated immediately since it is equal to the angle of incidence, 45°. The angle of refraction, θ_2, is obtained by substituting into Equation 4·67. Solving for θ_2, we obtain

$$\theta_2 = \sin^{-1}\left([\sin \theta_1]\frac{v_2}{v_1}\right)$$

However, v_1 is equal to v_0, the velocity in free space, and v_2 is equal to v_0 divided by the square root of k_e, as given in Equation 4·66. Solving, we find

$$\theta_2 = \sin^{-1}\left(\frac{v_0/\sqrt{4}}{v_0}\sin 45°\right)$$

$$\theta_2 = \sin^{-1}(0.354)$$

$$\theta_2 = 21° \qquad \textit{Ans. (a)}$$

The ratio of reflected to incident electric intensity is given by Equation 4·68:

$$\frac{E_3}{E_1} = \frac{\sin(45° - 21°)}{\sin(45° + 21°)} = \frac{\sin 24°}{\sin 66°}$$

$$\frac{E_3}{E_1} = 0.45 \qquad \textit{Ans. (b)}$$

4·6 WAVES IN CONDUCTING MEDIA

In a conducting medium, the conductance of the medium, σ, must be taken into account when the equations are solved. Solving the equations with the conductance results in the same form as the solution in free space except that the propagation constant, Γ, becomes complex. Similarly the characteristic impedance becomes complex. Γ being complex means that there will be attenuation as the wave is propagated and the **E** and **H** vectors will be out of time phase. Attenuation is to be expected since energy is lost by currents flowing in the medium. The propagation constant Γ is given by

$$\Gamma = \sqrt{j2\pi f k_m \mu_0 \sigma - (2\pi f)^2 k_e k_m \epsilon_0 \mu_0} \qquad (4\cdot70)$$

Similarly the intrinsic impedance, Z_0, obtained will be given by

$$Z_0 = \sqrt{\frac{j2\pi f k_m \mu_0}{j2\pi f k_e \epsilon_0 + \sigma}} \qquad (4\cdot71)$$

In both Equations 4·70 and 4·71, f is the frequency of the wave under consideration. An imperfect conductor is defined as a conductor where the conductivity, σ, is very much greater than $2\pi f k_e \epsilon_0$. Hence the displacement currents are negligibly small with respect to the conduction currents. With this simplification the propagation function, Γ, reduces to

$$\Gamma = \sqrt{\pi f k_m \mu_0 \sigma} + j\sqrt{\pi f k_m \mu_0 \sigma} \tag{4·72}$$

With this propagation factor the wave will decrease to $1/e$ of its original value when it has traveled a distance of $1/\sqrt{\pi f k_m \mu_0 \sigma}$ meters. This distance is referred to as the depth of penetration of the wave into the material.

EXAMPLE 4·5 Determine the depth of penetration of copper whose conductivity is 5.8×10^7 mhos per meter.

The depth of penetration is given by one over the real part of Γ in Equation 4·72. In the case of copper, k_m is equal to 1. Thus

$$\frac{1}{\sqrt{\pi f \mu_0 \sigma}} \quad \frac{1}{\sqrt{\pi \times 4\pi \times 10^{-7} \times 5.8 \times 10^7 \times f}}$$

$$\text{Depth} = \frac{0.066}{\sqrt{f}} \text{ meters} \qquad \textit{Ans.}$$

REFERENCE READING

G. W. PIERCE, *Electric Oscillations and Electric Waves*, New York, McGraw-Hill Book Co., 1920, Chapters 3, 4, 5, and 6 of Book II.

H. H. SKILLING, *Fundamentals of Electric Waves*, New York, John Wiley & Sons, 1942, Chapter X.

R. I. SARBACHER and W. A. EDSON, *Hyper and Ultrahigh Frequency Engineering*, New York, John Wiley & Sons, 1943, Chapters 3 and 4.

J A. STRATTON, *Electromagnetic Theory*, New York, McGraw-Hill Book Co., 1941, Chapter 5.

PROBLEMS

4·1 An electromagnetic wave in free space has an electric intensity with a maximum value of 2 millivolts per meter. Determine the value of Poynting's vector.

4·2 Find the relationship between E^+ and H^+ if k_e is equal to 2.5 and k_m is equal to 1.1.

4·3 Determine the direction of the reflected ray and the refracted ray when an electromagnetic wave in free space is incident at an angle of 50° on the surface of the material specified in problem 4·2.

4·4 Determine the magnitude of the reflected ray electric intensity vector in terms of the incident ray electric intensity vector in problem 4·3 if the magnetic vector is parallel to the reflecting surface.

4·5 Find the intensity of the electric and magnetic fields when power is being transmitted through space at 25 microwatts per square meter.

Chapter 5

RADIATION

5·1 Radiated energy

The existence of an electromagnetic wave capable of conveying power, as discussed in Chapter 4, introduces the problems of transmission, reception, and shielding—the often annoying problem of debarring it from circuits where it is not desired. Sound waves, in a very limited fashion, are utilized in an analogous manner. They are used to convey intelligible energy from one point in a medium to another, carrying the energy as a multiplicity of complex waves. These waves are, and we can use the same term, radiated from the human vocal organs or from mechanical sources such as loud speakers, tuning forks, and similar disturbances in the medium. The sound waves convey intelligence by means of their frequency, their amplitude, or combinations of both.

In radio the spectrum of electromagnetic wave frequencies available is very large and is growing continuously as our ability to use the higher frequencies improves. Many wave trains may be transmitted through the medium and be separated at the receiver; hence the waves do not interfere with one another. In sound, however, two sounds reaching the receiver, a human ear or a microphone, will interfere with one another. For this reason acoustical engineers carefully shield (better known as soundproof) their transmitting studios. A similar effect occurs in radio applications; electromagnetic energy must be kept from entering a circuit where it would interfere with the energy already in the circuit This, of course, is the well-known process of shielding, or arranging a circuit so that no extraneous energy is picked up.

The problems encountered in the transmission and reception of electromagnetic energy are: First, what type of radiator or antenna, as it is better known, is to be used and how does it function? Second, what are the characteristics of the antenna, its impedance, its directional pattern, its losses, and so on? Third, how is energy fed into the transmitting antenna? Fourth, what type of device is used to receive the energy (the receiving antenna)? And, fifth, what are the characteristics of the receiving antenna, the amount of energy it receives, its impedance, and so on? The problems are further complicated by the facts that the antennas are not usually known to exist in nature. It

is desired to create antennas that yield specific results. Not all the problems have been solved analytically, and the final results are a combination of mathematical analysis and practical tests.

The problem of shielding a circuit from electromagnetic waves is no less important. One solution is to use the directive properties of the transmitting element and the directive properties of the receiving element, combine them, and obtain the proper orientation of the circuits to prevent interaction. At the higher frequencies this solution is not usually possible and a metal shield is employed. What takes place is that the shield itself acts as a receiver and reradiates the energy it receives. This reradiated energy tends to cancel the direct wave energy penetrating the shield so that practically no foreign field is left within the shield. It is interesting to note that, similarly, any reradiation from a receiving antenna will affect the field surrounding it. A receiving antenna placed close enough to the transmitting antenna will cause the operating characteristics to change. As the distance between the receiving antenna and the transmitting antenna is increased, this mutual coupling effect grows smaller and smaller until its effect is negligible. How large the distances are for no interaction is determined from the electromagnetic calculations.

5·2 Polarized spherical wave

The electromagnetic field is generated by a distribution of charges and currents in a circuit. This electromagnetic field then interacts with the charges and currents in another circuit, causing the energy to be removed from the field. It is by this means that the energy is radiated and received. As mentioned in the previous chapter, at the origin of the electromagnetic field the wave expands in a spherical manner. Hence, to determine the radiation equations, it is desirable to study the polarized spherical wave.

Hertz used the following method of derivation in his book *Electric Fields*.* To obtain the equations for the polarized spherical wave obtained from an antenna, the boundary conditions for this type of wave are substituted into Maxwell's equations and the resultant equations solved. This is one of the standard methods followed in electromagnetic problems. In many cases, however, the resultant equations have not been solved and the answer has to be obtained by trial and error and judicious guesswork. Fortunately, however, the polarized spherical wave equations are capable of solution.

* See also G. W. Pierce's *Electric Oscillations and Electric Waves*, New York, McGraw-Hill Book Co., 1920, for a thorough treatment of this derivation.

For convenience, Maxwell's equations in the differential form are repeated below:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \tag{5·1a}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \tag{5·1b}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \tag{5·1c}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = i_x + \frac{\partial D_x}{\partial t} \tag{5·2a}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = i_y + \frac{\partial D_y}{\partial t} \tag{5·2b}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i_z + \frac{\partial D_z}{\partial t} \tag{5·2c}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho \tag{5·3}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{5·4}$$

A simple type of field to consider is the one caused by a current, I_z, flowing in a wire along the z axis at the origin. When this is used there will be no z component of **H** since there can be no component created parallel to the current flow. The exact form of this current and the effect of the discontinuity of the medium at that point is not yet specified. The medium everywhere else is assumed to be homogeneous. In mediums where radiation takes place the charge density is nearly always equal to zero. It is also assumed, therefore, that the charge density everywhere throughout the medium is equal to zero except at the origin. This assumption is true when radiation takes place in free space. Thus the equations can be further simplified by having the radiation take place in free space. The only added factors that have to be taken into account are that the relative dielectric constant, k_e, and the relative permeability constant, k_m, are both equal to one. Gathering together all the boundary conditions, we obtain

$$
\begin{aligned}
H_z &= 0 \\
\rho &= 0 \text{ (except at the origin)} \\
I_x &= 0 \text{ (at the origin)} \\
I_y &= 0 \text{ (at the origin)} \\
i_x &= 0 \\
i_y &= 0 \\
i_z &= 0 \text{ (except at the origin)} \\
k_e = k_m &= 1
\end{aligned}
\tag{5·5}
$$

I_x and I_y are the currents which may flow at the origin and must not be confused with the conduction current densities, i_x, i_y, and i_z, which are everywhere zero in free space except at the origin where the current exists. In other words, the type of field to be analyzed was so chosen that when the coordinates are aligned correctly there will be no z component of **H** throughout the medium. This is accomplished by having I_x and I_y always zero, as indicated in Equations 5·5. The type of oscillator at the origin will be determined later.

The next step in the solution calls for substituting the boundary conditions, as given in Equations 5·5, into Maxwell's equations, given in one form in Equations 5·1 to 5·4. First the condition that H_z is equal to zero is substituted into the equation for the z component of the curl of **E**, given in Equation 5·1*c*.

$$
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \tag{5·6}
$$

Equation 5·6 is a well-known type of differential equation wherein the existence of a scalar function is implied. Calling this scalar function of position G, we find that E_x and E_y are defined by

$$
\begin{aligned}
E_x &= -\frac{\partial G}{\partial x} \\
E_y &= -\frac{\partial G}{\partial y}
\end{aligned}
\tag{5·7}
$$

When E_x and E_y, as given in Equations 5·7, are substituted into Equation 5·6, it should yield an identity. This is valid because the second derivative of G taken first with respect to y and then with respect to x is equal to the second derivative of G taken first with respect to x and then with respect to y. This, of course, is also correct for partial derivatives.

The boundary condition that ρ is equal to zero can now be substituted into Equation 5·3. Using, in addition, the relationships given

in Equations 5·7, we find that the divergence equation becomes

$$-\frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial y^2} + \frac{\partial E_z}{\partial z} = 0 \tag{5·8}$$

Solving for E_z, we obtain

$$E_z = \int \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \partial z \tag{5·9}$$

Equations 5·7 and 5·9 express the components of **E** in terms of the derivatives of the scalar function G. To solve Equation 5·9, we must obtain the function whose partial derivative with respect to z will be equal to the function G. Calling this new function F, we evaluate G by

$$G = -\frac{\partial F}{\partial z} \tag{5·10}$$

This value will have to be true if Equation 5·9 is capable of solution. The function F, when it is obtained, will be sufficient to define all the components of the electric intensity. To secure the expression for E_z in terms of the scalar function F, we first substitute the equation for G from Equation 5·10 into Equation 5·9 and integrate with respect to z:

$$E_z = -\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} \tag{5·11}$$

The constant of integration is neglected because it would define a component of **E** which would not vary with position. A component of this type cannot exist in a traveling electromagnetic wave. To solve for the scalar function F, it is desirable to transform the equation for E_z into a simplified form. This is accomplished by first adding and subtracting the second derivative of F with respect to z from the right-hand side of Equation 5·11

$$E_z = \frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial z^2} \tag{5·12}$$

A new operator, ∇^2, is now introduced. As was discussed previously, the square of the magnitude of a vector is obtained by taking the dot product of the vector and itself. Similarly, the symbol ∇^2 can be thought of as representing the dot product of the vector operator nabla, ∇, and itself. The actual use of the term "magnitude" with respect to the operator nabla is meaningless and should not be employed. However, it is used *symbolically* in conjunction with this new operator for

convenience; actually, it is used because of the similarity between the appearance of the result and the dot product. It is called the "Laplacian" after the French mathematician and is defined as follows:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{5·13}$$

It is used like a scalar and may be utilized as a multiplying factor of either a vector or a scalar. There is no cross or dot between ∇^2 and the vector or scalar it is multiplying. When used in conjunction with a scalar, it signifies the sum of the second derivative of the scalar with respect to x plus the second derivative of the scalar with respect to y plus the second derivative of the scalar with respect to z. The result of course is also a scalar. When used in conjunction with a vector it also signifies the sum of the three second derivatives of the vector, with respect to x, with respect to y, and with respect to z. The result is another vector.

Examining the right-hand side of Equation 5·12, we find that its last three terms are equal to the Laplacian of F, expressed as $\nabla^2 F$. Substituting the Laplacian of F into Equation 5·12 and the expression for G as given in Equation 5·10 into Equation 5·7, we obtain the three components of **E** as functions of the scalar variable F:

$$\begin{aligned} E_x &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right) \\ E_y &= \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}\right) \\ E_z &= \frac{\partial^2 F}{\partial z^2} - \nabla^2 F \end{aligned} \tag{5·14}$$

To completely define the field in terms of the scalar function of position, F, the components of **H** have to be determined as functions of this scalar. The boundary conditions of Equations 5·5 are now substituted into the curl equations, Equations 5·2a and 5·2b:

$$\begin{aligned} -\frac{\partial H_y}{\partial z} &= \epsilon_0 \frac{\partial E_x}{\partial t} \\ \frac{\partial H_x}{\partial z} &= \epsilon_0 \frac{\partial E_y}{\partial t} \end{aligned} \tag{5·15}$$

E_x and E_y have already been expressed in terms of F, Equation 5·14.

Replacing these two terms in Equations 5·15, we get

$$
\begin{aligned}
-\frac{\partial H_y}{\partial z} &= \epsilon_0 \frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right)\right] \\
\frac{\partial H_x}{\partial z} &= \epsilon_0 \frac{\partial}{\partial t}\left[\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}\right)\right]
\end{aligned}
\tag{5·16}
$$

These two equat ons may be integrated to obtain the two existing components of the magnetic field intensity. The constant of 'ntegration is again equated to zero 'nasmuch as no constant term exists in the traveling wave. From the boundary conditions stated in Equations 5·5, the z component of **H** is always zero; hence the three components of the field intensity **H** are

$$
\begin{aligned}
H_x &= \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y}\right) \\
H_y &= -\epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x}\right) \\
H_z &= 0
\end{aligned}
\tag{5·17}
$$

Equations 5·14 and 5·17 express all the components of the electric and magnetic field intensity vectors of the spherical field in terms of the derivatives of only one scalar variable, F. All that remains is to obtain the solution for F as well as its relationship to the method of wave generation and the equations will be solved. Actually, the solution desired is the relationship of this function, F, to the distribution of currents and charges in the generating circuit at the origin.

Solving for F, we must return to Maxwell's equations and use those equations which have not yet been employed in the derivation of Equations 5·14 and 5·17. First, the expressions for E_y and E_z as given in Equations 5·14 are substituted into Equation 5·1a:

$$
-\mu_0 \frac{\partial H_x}{\partial t} = \frac{\partial}{\partial y}\left(\frac{\partial^2 F}{\partial z^2}\right) - \frac{\partial}{\partial y}\left(\nabla^2 F\right) - \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z}\right)\right]
\tag{5·18}
$$

However, from the theory of differential calculus, the partial derivative with respect to y of the second partial derivative with respect to z of the function F is equal to the partial derivative with respect to z, then with respect to y and then with respect to z of the function F. In other words, the order in which the derivatives are taken does not affect the result. Thus, simplifying Equation 5·18, we obtain

$$
\mu_0 \frac{\partial H_x}{\partial t} = \frac{\partial}{\partial y}\left(\nabla^2 F\right)
\tag{5·19}
$$

However, H_x is given in terms of F in Equations 5·17. Replacing H_x

by its equivalent value from Equations 5·17, we get

$$\mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}(\nabla^2 F) \tag{5·20}$$

This is an equation in F only. If it can be solved, the field will be completely defined. Again, the order in which the derivatives are taken will not affect the result so that

$$\mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu_0\epsilon_0 \frac{\partial^2 F}{\partial t^2}\right) \tag{5·21}$$

Hence both sides of Equation 5·20 may be integrated with respect to y. The integration constants may be disregarded without any loss in generality:

$$\mu_0\epsilon_0 \frac{\partial^2 F}{\partial t^2} = \nabla^2 F \tag{5·22}$$

which is an equation in F only since μ_0 and ϵ_0 are constants.

To proceed with the solution of Equation 5·22 it is desirable to look into the type of wave which it will specify. Since it was shown that the plane electromagnetic wave solution of Maxwell's equations specifies a traveling type of wave made up of two waves traveling in opposite directions, a good approach would be to see if this type of solution would satisfy Equation 5·22.

In a spherical wave the equiphase surfaces are spherical. In the electromagnetic wave they are spreading out from the origin; hence the origin is the center of all the spherical equiphase surfaces. Let s be the distance from any point in the coordinate space to the origin. In other words, s is a radial line. Since this spherical wave originates at the origin and travels outward, a logical assumption would be that the wave is a function of s. The distance, s, in terms of the coordinates of a point, x, y, and z, is given by

$$s = \sqrt{x^2 + y^2 + z^2} \tag{5·23}$$

The wave will also be a function of the time, t. The assumption will be made now that the scalar function of position in space, the function F, is a function of the distance, s, and the time, t. Calling this new function $f(s, t)$, we assume that F is equal to $f(s, t)$. Letting F be equal to this new function indicates that F may be completely expressed in terms of the two variables, s and t. We shall have to prove that such a function is a solution of Equation 5·22 as well as determine the function itself.

Notice that s is a function of x, y, and z. All the derivatives with respect to x, y, or z of the function F now have to be changed to derivatives of the function with respect to s. The partial derivative of F with respect to x is given by

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial s}\frac{\partial s}{\partial x} \tag{5·24}$$

where

$$\frac{\partial s}{\partial x} = \frac{\partial(\sqrt{x^2 + y^2 + z^2})}{\partial x} = \frac{x}{s} \tag{5·25}$$

Substituting Equation 5·25 into Equation 5·24, we obtain the derivative of the function F with respect to x:

$$\frac{\partial F}{\partial x} = \frac{x}{s}\frac{\partial F}{\partial s} \tag{5·26}$$

However, equation 5·22 contains only the second derivatives of the function. To obtain the second derivative of F with respect to x, Equation 5·26 is differentiated with respect to x. Equation 5·26 may be considered to consist of the product of three functions, x, $1/s$, and $\partial F/\partial s$. In other words, each one of these is a separate function of x and the right-hand side of Equation 5·26 is the product of the three. The derivative of such a product is equal to the sum of three terms, each term being the derivative of one of the functions multiplied by the remaining two functions. It is merely an expansion of the derivative of the product of two functions. The derivative of Equation 5·26, the second partial derivative of F with respect to x, is expressed by

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{s}\frac{\partial F}{\partial s} - \frac{x}{s^2}\frac{\partial s}{\partial x}\frac{\partial F}{\partial s} + \frac{x}{s}\frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial s^2} \tag{5·27}$$

This may be simplified by using Equation 5·25.

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{s}\frac{\partial F}{\partial s} - \frac{x^2}{s^3}\frac{\partial F}{\partial s} + \frac{x^2}{s^2}\frac{\partial^2 F}{\partial s^2} \tag{5·28}$$

In the expression for s, as given in Equation 5·23, we see that s is exactly the same function of y and z as of x. Hence the second partial derivatives of F with respect to y and z can be written down directly by substituting the new variable for x everywhere that x appears in the solution as given by Equation 5·28. Thus

$$\begin{aligned} \frac{\partial^2 F}{\partial y^2} &= \frac{1}{s}\frac{\partial F}{\partial s} - \frac{y^2}{s^3}\frac{\partial F}{\partial s} + \frac{y^2}{s^2}\frac{\partial^2 F}{\partial s^2} \\ \frac{\partial^2 F}{\partial z^2} &= \frac{1}{s}\frac{\partial F}{\partial s} - \frac{z^2}{s^3}\frac{\partial F}{\partial s} + \frac{z^2}{s^2}\frac{\partial^2 F}{\partial s^2} \end{aligned} \tag{5·29}$$

These second derivatives, Equations 5·28 and 5·29, enter into Equation 5·22 in the form of $\nabla^2 F$, $\nabla^2 F$ being equal to the sum of the three second derivatives. Taking the sum and substituting s^2 for $(x^2 + y^2 + z^2)$, we obtain

$$\nabla^2 F = \frac{2}{s}\frac{\partial F}{\partial s} + \frac{\partial^2 F}{\partial s^2} \tag{5·30}$$

This expression can be further simplified by recognizing the relationship that the right-hand side of the equation is the second partial derivative of the product of s and F divided by s. To prove this, the first derivative is obtained—

$$\frac{\partial (sF)}{\partial s} = F + s\frac{\partial F}{\partial s} \tag{5·31}$$

and the second partial derivative is expressed by

$$\frac{\partial^2 (sF)}{\partial s^2} = 2\frac{\partial F}{\partial s} + s\frac{\partial^2 F}{\partial s^2} \tag{5·32}$$

When these are substituted into Equation 5·30, the resultant expression for the Laplacian of F is given by

$$\nabla^2 F = \frac{1}{s}\frac{\partial^2 (sF)}{\partial s^2} \tag{5·33}$$

Thus $\nabla^2 F$ is expressed in terms of the variable F, a function of s and t, and the variable s itself. Actually sF can be considered to be a single function of s and t, as will be shown later.

The Laplacian of F, as defined by Equation 5·33, can be substituted into the equation to be solved, namely, Equation 5·22:

$$\mu_0\epsilon_0 \frac{\partial^2 F}{\partial t^2} = \frac{1}{s}\frac{\partial^2 (sF)}{\partial s^2} \tag{5·34}$$

This equation is not completely satisfactory inasmuch as there are two functions involved, F and sF. It can be rectified by moving the s, which is not a function of t, from the right-hand side of Equation 5·34 over to the left-hand side and incorporating it into the time derivative:

$$\mu_0\epsilon_0 \frac{\partial^2 (sF)}{\partial t^2} = \frac{\partial^2 (sF)}{\partial s^2} \tag{5·35}$$

The quantity sF can now be considered to be a single variable. Let us compare this equation with Equations 4·7 and 4·8, which are the equations solved in the plane wave case. We notice the similarity. Hence the assumptions were correct and the solution of Equation 5·35 will be

the sum of two exponential functions, one being a wave traveling in the plus s direction and the other a wave traveling in the minus s direction. Continuing the analogy between this solution and the plane wave case, we notice that the result will involve, in a harmonically generated wave, the term ω, which is equal to $2\pi f$, where f is the frequency of generation, and β, the term which determines the phase velocity through the medium. Thus

$$sF = g(\omega t - \beta s) + h(\omega t + \beta s) \tag{5·36}$$

where $g(\omega t - \beta s)$ represents an exponential phasor function of $(\omega t - \beta s)$ defining a wave traveling in the plus s direction. In other words, $(\omega t - \beta s)$, preceded by a j term, will appear in the solution as the exponent of e, defining the solution as a traveling wave. It does not mean that the rest of the solution is independent of s and t; it is not as will be shown. Similarly $h(\omega t - \beta s)$ represents an exponential function of $(\omega t - \beta s)$ defining a wave traveling in the minus s direction. Let us glance back now at Equations 5·14 and 5·17 and notice that F, when it is obtained, will completely define the electromagnetic field but is, in itself, a scalar position of space. We must not confuse the function F with the vectors $\mathbf{E}$ and $\mathbf{H}$.

The functions g and h, if they are sinusoidal, can be expressed as functions of two amplitude factors, K^- and K^+, as follows:

$$\begin{aligned} g(\omega t - \beta s) &= K^+ e^{j(\omega t - \beta s)} \\ h(\omega t + \beta s) &= K^- e^{j(\omega t + \beta s)} \end{aligned} \tag{5·37}$$

where K^+ and K^- are boundary condition constants. These equations follow from the analogy with the plane electromagnetic wave. We notice the absence of an attenuation factor; it is left out inasmuch as the medium is assumed to be free space. The phase constant, β, may be obtained by substituting either equation of Equations 5·37 into Equation 5·35. As in the plane electromagnetic wave, β comes out to be equal to $\omega\sqrt{\mu_0\epsilon_0}$.

The functions may be expressed in terms of the velocity of propagation of the wave, v, by substituting for β in the exponential where, from the plane wave solution, v is equal to $1/\sqrt{\mu_0\epsilon_0}$. Thus

$$\omega t - \beta s = \omega t - \omega\sqrt{\mu_0\epsilon_0}\, s$$

$$\omega t - \beta s = \omega(t - \sqrt{\mu_0\epsilon_0}\, s)$$

$$\omega t - \beta s = \omega\left(t - \frac{s}{v}\right) \tag{5·38}$$

Similarly,

$$\omega t + \beta s = \omega\left(t + \frac{s}{v}\right) \tag{5·39}$$

The function F can now be expressed as a function of $\left(t - \frac{s}{v}\right)$ plus a function of $\left(t + \frac{s}{v}\right)$. Incorporating ω into the constants and dividing through by s, we obtain

$$F = \frac{1}{s}\,g\left(t - \frac{s}{v}\right) + \frac{1}{s}\,h\left(t + \frac{s}{v}\right) \tag{5·40}$$

Assuming now that the generator is located in free space with no reflecting surfaces present, we can equate the h function to zero. The reflected wave can usually be determined by the method discussed under the topics of reflected and refracted waves in Chapter 3. As will be shown later, if the reflecting surface is not comparatively close to the generator, it will not affect the generator and therefore not affect the g function. Thus, to solve for the g function in terms of the distribution of currents and charges in the generator circuit, the h function is equated to zero. Hence

$$F = \frac{1}{s}\,g\left(t - \frac{s}{v}\right) \tag{5·41}$$

Equation 5·50 is now substituted into the expressions for **E** and **H** as given in Equations 5·14 and 5·17. From the result, the function $g\left(t - \frac{s}{v}\right)$ will be obtained in relation to the generator so that **E** and **H** can be calculated. For convenience, let g represent $g\left(t - \frac{s}{v}\right)$ so that everywhere g is indicated it will represent a function. It is done to simplify the equations as much as possible without modifying their meanings. It should be noted that the derivative of g with respect to $\left(t - \frac{s}{v}\right)$ is equal to the derivative of g with respect to t. The derivatives necessary to substitute Equation 5·41 into the equations for **E** and **H** are

$$
\begin{aligned}
\frac{\partial g}{\partial s} &= -\frac{1}{v}\frac{\partial g}{\partial t} \\
\frac{\partial F}{\partial z} &= -\frac{z}{s^3}g - \frac{z}{s^2 v}\frac{\partial g}{\partial t} \\
\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z}\right) &= \frac{xz}{s^3}\left(\frac{3g}{s^2} + \frac{3}{sv}\frac{\partial g}{\partial t} + \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2}\right) \\
\frac{\partial^2 F}{\partial z^2} &= \left(\frac{3z^2}{s^5} - \frac{1}{s^3}\right)g + \left(\frac{3z^2}{s^4 v} - \frac{1}{s^2 v}\right)\frac{\partial g}{\partial t} + \frac{z^2}{s^3 v^2}\frac{\partial^2 g}{\partial t^2} \\
\nabla^2 F &= \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = \frac{1}{sv^2}\frac{\partial^2 g}{\partial t^2}
\end{aligned}
\tag{5·42}
$$

Substituting Equations 5·42 into the equations for the components of **E** as stated in Equations 5·14, we obtain

$$
\begin{aligned}
E_x &= \frac{xz}{s^3}\left(\frac{3g}{s^2} + \frac{3}{sv}\frac{\partial g}{\partial t} + \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2}\right) \\
E_y &= \frac{yz}{s^3}\left(\frac{3g}{s^2} + \frac{3}{sv}\frac{\partial g}{\partial t} + \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2}\right) \\
E_z &= \frac{2}{s^2}\left(\frac{g}{s} + \frac{1}{v}\frac{\partial g}{\partial t}\right) - \frac{x^2 + y^2}{s^3}\left(\frac{3g}{s^2} + \frac{3}{sv}\frac{\partial g}{\partial t} + \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2}\right)
\end{aligned}
\tag{5·43}
$$

Substituting Equations 5·42 also into Equations 5·17 for the components of **H**, we get

$$
\begin{aligned}
H_x &= -\frac{\epsilon_0 y}{s^2}\left(\frac{1}{s}\frac{\partial g}{\partial t} + \frac{1}{v}\frac{\partial^2 g}{\partial t^2}\right) \\
H_y &= \frac{\epsilon_0 x}{s^2}\left(\frac{1}{s}\frac{\partial g}{\partial t} + \frac{1}{v}\frac{\partial^2 g}{\partial t^2}\right) \\
H_z &= 0
\end{aligned}
\tag{5·44}
$$

Equations 5·43 and 5·44 express the electromagnetic field in terms of the function g.

Since the field is spherical, it is best to transform the equations into spherical coordinates. In Figure 5·1 we see the rectangular and spherical coordinates of the point x, y, z. The spherical coordinates are: ρ, the distance from the point in question to the origin; θ, the angle that a line from the origin to the point makes with the positive segment of the z axis; and ϕ, the angle between the projection of the line on the z plane through the origin and the positive segment of the x axis. By transforming the equations to these coordinates the implications of the equations become more evident.

Let us allow r to be equal to the length of the projection on the z plane of a line from the origin to the point x, y, z. Thus

$$r = \sqrt{x^2 + y^2} \tag{5·45}$$

The distance ρ will be given by

$$\rho = \sqrt{x^2 + y^2 + z^2} = s \tag{5·46}$$

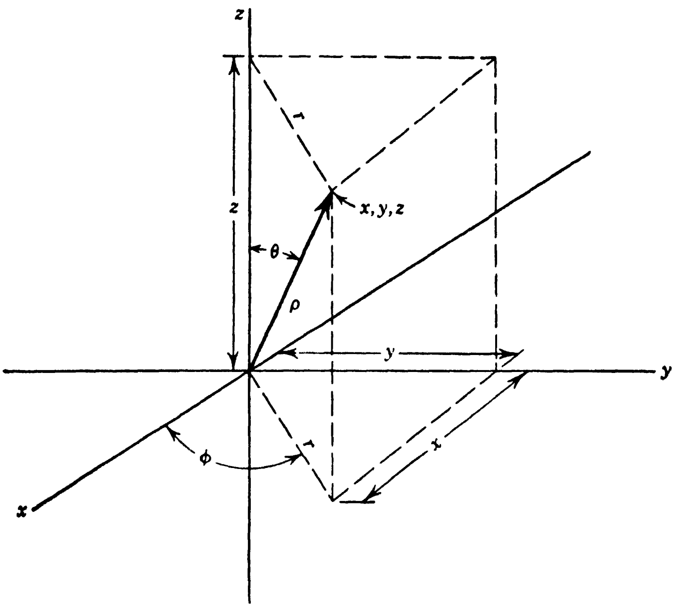


FIG. 5·1 Spherical coordinates showing a vector of magnitude ρ as well as its components in rectangular coordinates.

The components of **E** and **H** are now transformed from the x, y, and z components of rectangular coordinates to the ρ, θ, and ϕ components of spherical coordinates. The ρ component is that component taken in the ρ direction which, in these coordinates, is the radial direction; hence the ρ component is the radial component. The θ component is taken in the direction determined by a differential change in θ; hence the θ component is perpendicular to the radial line and in the plane containing the point in question and the z axis. The ϕ component similarly is in the direction determined by a differential change in the angle ϕ; consequently, it is also perpendicular to the radial line but is parallel to the z plane. As in rectangular coordinates, these three components will completely define a vector. The magnitude of the com-

ponents are determined by the projections of the vector on lines extending through the base of the vector in the direction of the components.

Transforming first the components of **H**, we obtain for H_ϕ

$$H_\phi = H_y \frac{x}{r} - H_x \frac{y}{r} \tag{5·47}$$

Substituting H_x and H_y from Equation 5·44 into Equation 5·47 and remembering that r/ρ is equal to $\sin\theta$, we get

$$H_\phi = \epsilon_0 \frac{\sin\theta}{\rho}\left(\frac{1}{\rho}\frac{\partial g}{\partial t} + \frac{1}{v}\frac{\partial^2 g}{\partial t^2}\right) \tag{5·48}$$

The θ and ρ components of **H** come out to be zero, a very convenient result inasmuch as it means that the magnetic intensity vector is completely defined by a single component. Thus the direction of the magnetic field is immediately tied down to the ϕ direction, and we find that a magnetic line of force will be a circle in a z plane with the z axis at its center.

The components of **E** are now obtained in the same manner. The ρ component of **E** is given by

$$E_\rho = E_x \frac{x}{\rho} + E_y \frac{y}{\rho} + E_z \frac{z}{\rho} \tag{5·49}$$

This equation is standard inasmuch as the radial component is equal to the sum of rectangular components multiplied by their respective direction cosines. Replacing the rectangular components by their equivalents from Equations 5·43 and noticing that z/ρ is equal to $\cos\theta$, we obtain

$$E_\rho = \frac{2\cos\theta}{\rho^2}\left(\frac{g}{\rho} + \frac{1}{v}\frac{\partial g}{\partial t}\right) \tag{5·50}$$

Referring now to Figure 5·1, we see that the θ component is given by

$$E_\theta = E_r \cos\theta - E_z \sin\theta \tag{5·51}$$

where E_r is used only as a convenient means of obtaining the solution. Actually there is no need for such a component except for visual understanding. It is equal to the following:

$$E_r = E_x \cos\theta + E_y \sin\theta \tag{5·52}$$

Substituting Equation 5·52 into Equation 5·51 and then replacing the rectangular components from Equation 5·43, we obtain

$$E_\theta = \frac{\sin\theta}{\rho}\left(\frac{g}{\rho^2} + \frac{1}{v\rho}\frac{\partial g}{\partial t} + \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2}\right) \tag{5·53}$$

The ϕ component of **E** comes out to be zero. Gathering together all the spherical components of the electric and magnetic field intensities of the spherical electromagnetic wave, we have

$$\begin{aligned} H_\phi &= \epsilon_0 \frac{\sin\theta}{\rho}\left(\frac{1}{\rho}\frac{\partial g}{\partial t} + \frac{1}{v}\frac{\partial^2 g}{\partial t^2}\right) \\ H_\theta &= 0 \\ H_\rho &= 0 \end{aligned} \tag{5·54}$$

$$\begin{aligned} E_\phi &= 0 \\ E_\theta &= \frac{\sin\theta}{\rho}\left(\frac{g}{\rho^2} + \frac{1}{\rho v}\frac{\partial g}{\partial t} + \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2}\right) \\ E_\rho &= \frac{2\cos\theta}{\rho^2}\left(\frac{g}{\rho} + \frac{1}{v}\frac{\partial g}{\partial t}\right) \end{aligned} \tag{5·55}$$

These two sets of equations completely define the field around a generator and meet the boundary conditions set forth in Equations 5·5. The type of generator is as yet unknown.

5·3 The doublet as a spherical field generator

For Equations 5·54 and 5·55 to be useful they should be linked, through the function g, to a practical and usable generator. Because it is so common in nature and because it is easy to obtain and is nearly always used in electrical circuits the resultant field will be assumed to vary sinusoidally with time.

Examining Equations 5·54 and 5·55, the equations for the field, we notice that the function g is the only item that varies with time. This function enters into the equations either directly or as a derivative with respect to time. If a function is sinusoidal with respect to time, all its derivatives will be sinusoidal, the time equation being shifted in phase 90° with each derivative. It is for this reason that g is considered to be a sinusoidal function, the real part of an exponential with an exponent of $j\omega t$. Again, ω is the angular velocity of the phasor, equal to $2\pi f$, where f is the frequency of rotation and the frequency of the wave.

Let us consider the case wherein

$$\frac{1}{v}\left|\frac{\partial g}{\partial t}\right| \ll \frac{1}{\rho}\left|g\right| \tag{5·56}$$

This means that ρ is relatively small. However, with g a sinusoidal

function as has been assumed, the derivative of g with respect to time is equal to $j\omega g$, inasmuch as the derivative of an exponential function is the original function multiplied by the derivative of the exponent. Substituting this result into Equation 5·57, we obtain

$$\frac{\omega}{v} \ll \frac{1}{\rho} \tag{5·57}$$

However ω is equal to $2\pi f$ and v is equal to the wavelength λ times f. Using these relationships, we find that the frequency cancels out. Inverting both sides of the equation, which also means that the inequality signs will have to be reversed, we obtain

$$\rho \ll \frac{\lambda}{2\pi} \tag{5·58}$$

Thus Equation 5·56 indicates that ρ, the radial distance from the generator, is assumed to be very small with respect to a wavelength of the wave under consideration.

To investigate the field as close as this to the generator it is advisable to examine Equations 5·14. The equation for E_z involves the Laplacian of F. Remembering that s is equal to ρ and referring to the last two equations of Equations 5·42 we see that $\nabla^2 F$ is negligible with respect to $\partial^2 F/\partial z^2$. From Equation 5·14 at this distance from the generator, E_x, E_y, and E_z are the x, y, and z derivatives of the same quantity $\partial F/\partial z$. Hence the electric force near the origin has the form of an ordinary static potential ψ, where

$$\psi = \frac{\partial F}{\partial z} \tag{5·59}$$

The derivative of F with respect to z is given in the second equation of Equations 5·42. The second term, which is negligibly small, may be dropped and the expression for ψ becomes

$$\psi = -\frac{z}{\rho^3} g \tag{5·60}$$

It will be shown now that Equation 5·60 is exactly the same potential as that obtained from a doublet at the origin whose moment is $\epsilon_0 g$, provided its length is negligible in comparison with 2ρ.

A doublet consists of two opposite charges separated by a definite distance as shown in Figure 5·2. In this case the charges are $+q$ and $-q$, as shown, and separated by a distance d. They are lying along the z axis with the origin midway between them. The electrostatic

potential, ψ_0, at a point P, which is ρ distance from the origin, is given by

$$\psi_0 = \frac{q}{4\pi\epsilon_0\rho_1} - \frac{q}{4\pi\epsilon_0\rho_2} \tag{5·61}$$

The 4π enters into the equations because the M.K.S. system of units is used throughout. However, ρ_1 can be taken as equal to the mean dis-

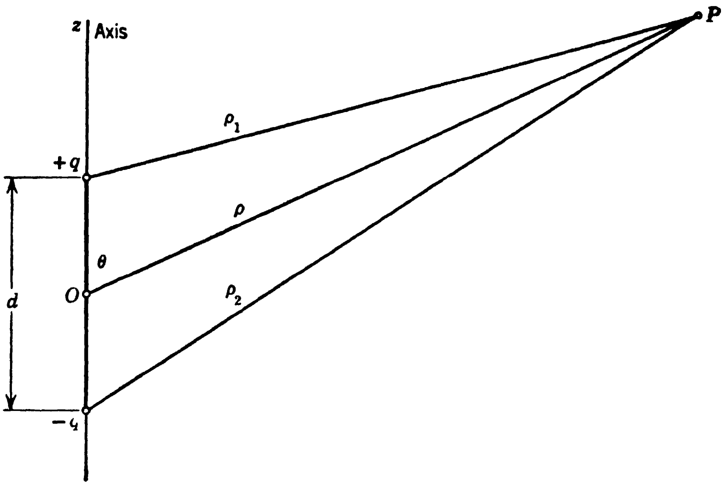


FIG. 5·2 A doublet at the origin oriented in the z direction and its dimensions with respect to the point P in space. The charge at one instant of time is shown as $-q$ and $+q$.

tance ρ minus (d cos θ) over 2, and ρ_2 as equal to ρ plus the same factor —provided, as stated above, the distance ρ is large with respect to d. Expressing Equation 5·61 in terms of the mean distance ρ and simplifying, we obtain

$$\psi_0 = \frac{qd\cos\theta}{4\pi\epsilon_0\left[\rho^2 - \frac{(d\cos\theta)^2}{4}\right]} \tag{5·62}$$

The second term in the brackets in the denominator has a maximum value of $d^2/4$, which, with the assumption that d is negligible with respect to 2ρ, drops out. Putting in the equality that cos θ is equal to to z/ρ, we find that the expression for ψ_0 becomes

$$\psi_0 = \frac{z}{\rho^3}\frac{qd}{4\pi\epsilon_0} \tag{5·63}$$

Now let us compare the result obtained here with the expression for ψ obtained in Equation 5·60. They are similar if

$$\frac{qd}{4\pi\epsilon_0} = g \tag{5·64}$$

This result now establishes that a minute doublet can create a field similar to that obtained in the spherically polarized case for very small values of ρ. The value of the function g for the dipole is given in Equa-

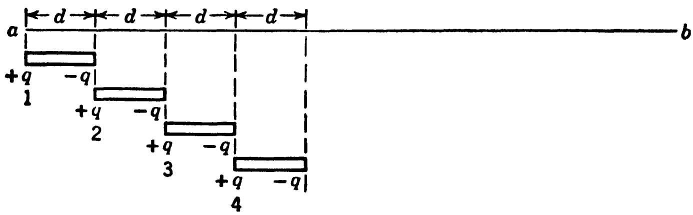


FIG. 5·3 Doublets and charges in a long wire broken up into segments. Dotted lines connect each segment with its actual position in the wire ab.

tion 5·64 in terms of the charge q, the distance d, and the constants π and ϵ_0. Thus we have the interesting conclusion that the spherically polarized field, with the boundary conditions given in Equations 5·5, may be generated by a doublet of very small length. This conclusion has proven to be true and calculations made from it have proven to be correct.

To be useful, the doublet case has to be extended to take in currents flowing in a wire. In Figure 5·3 is shown a wire broken up into small segments, 1, 2, 3, and so on. Each of these segments, when removed from the wire, is assumed to have equal and opposite charges on its ends, as shown. Thus each one of these segments is a small doublet whose length can be made as small as desired. When the doublets are connected end to end to form the original wire, all the charges on the ends which touch will tend to neutralize one another. Those on the very end can be assumed to be connected across the terminals of a generator. Thus the wire could be considered to be made up of very many minute doublets connected end to end.

Before the expressions for g can be substituted into the equations, it is desirable to determine the derivatives found in the solution. One is the partial derivative of the doublet moment, qd (as given in Equation 5·64), with respect to time, t:

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial t}\left(\frac{qd}{4\pi\epsilon_0}\right) \tag{5·65}$$

However, everything in the equation except q is constant with time so that Equation 5·65 reduces to

$$\frac{\partial g}{\partial t} = \frac{d}{4\pi\epsilon_0} \frac{\partial q}{\partial t} \tag{5·66}$$

But the derivative, with respect to time, of the charge q is equal to the current, i, flowing in the wire at the doublet under consideration. Thus the derivative, with respect to time, of the function g may be expressed in terms of the current. Allowing the length of the doublet to be the differential distance, ds, we obtain

$$\frac{\partial g}{\partial t} = \frac{i}{4\pi\epsilon_0} ds \tag{5·67}$$

The second partial derivative of g with respect to time can be written down directly since it only involves taking the partial dervative of the current, i:

$$\frac{\partial^2 g}{\partial t^2} = \frac{ds}{4\pi\epsilon_0} \frac{\partial i}{\partial t} \tag{5·68}$$

There are now enough terms to obtain the field components of the electromagnetic field in terms of the current and charges on a small segment of length, ds, situated in a wire carrying current. The complete equations, including the exponential factor $e^{-j\beta\rho}$, contained in g, therefore, are

$$\begin{aligned} dH_\phi &= \left[\epsilon_0 \frac{\sin\theta}{\rho}\left(\frac{i}{4\pi\epsilon_0\rho} + \frac{1}{4\pi\epsilon_0 v}\frac{\partial i}{\partial t}\right)\right] e^{-j\beta\rho} ds \\ dH_\theta &= 0 \\ dH_\rho &= 0 \end{aligned} \tag{5·69}$$

$$\begin{aligned} dE_\phi &= 0 \\ dE_\theta &= \left[\frac{\sin\theta}{\rho}\left(\frac{q}{4\pi\epsilon_0\rho^2} + \frac{i}{4\pi\epsilon_0\rho v} + \frac{1}{4\pi\epsilon_0 v^2}\frac{\partial i}{\partial t}\right)\right] e^{-j\beta\rho} ds \\ dE_\rho &= \left[\frac{2\cos\theta}{\rho^2}\left(\frac{q}{4\pi\epsilon_0\rho} + \frac{i}{4\pi\epsilon_0 v}\right)\right] e^{-j\beta\rho} ds \end{aligned} \tag{5·70}$$

The reason the differentials of the field components are used in the equations is that, actually, these equations express only the field caused by a small differential length of the radiating wire; to obtain the complete field, the equations have to be integrated over the whole length

of wire concerned. The foregoing equations represent, however, the complete field generated by a minute doublet of length ds.

5·4 The induction field

Equations 5·69 and 5·70 are complete and represent the field around the generator both close to the generator and distant from it. The laws for the region close to the generator are the simple laws of electrostatics and magnetostatics. This close region, sometimes called the near zone, is defined by Equation 5·58, the distance from the generator (or wire as has been chosen) being less than the wavelength under consideration divided by 2π.

It has already been shown in the derivation of the doublet equations how the equations reduce to the ordinary electrostatic equations. This does not mean that the other portions of the equations do not exist but rather that those portions are so small in the near region that compared to the other terms they may be neglected without any appreciable effect on the results. As soon as distances are involved which do not meet the limitations for the near zone, the complete equations have to be employed unless it is proven that other portions of the equations may be neglected.

In the near region, the distances are small with respect to a wavelength; hence the time involved, which is equal to $\omega s/v$, is so minute that the effects may be considered to be felt instantaneously throughout the near zone. It is not really instantaneous but it is so very close to instantaneous that making this assumption will not influence any of the results detectably. Of course this assumption leads to an infinite electromagnetic velocity which, although it is known not to be absolutely true, is acceptable for calculations. Very often, therefore, the near region is also referred to as the region where an infinite velocity of propagation of electromagnetic effects is assumed.

Examining now the equation for the magnetic field intensity with the limitation of the near region, we find the second term to be negligible. The magnetic field is therefore completely specified by

$$dH_\phi = i\frac{\sin\theta}{4\pi\rho^2}\,ds \tag{5·71}$$

This is the equation used in calculating the field in circuits at low frequencies where only the induction field is involved. Again, because the distances are very much smaller than a wavelength, the action is assumed to be instantaneous.

We notice how the size of the region in which the induction field can be used is dependent on both the actual distances involved and the

frequency. A circuit that at one megacycle can be treated throughout with induction field equations may show completely extraneous results if the same equations are employed for the same circuit at 100 megacycles. Of course at 10,000 megacycles there are very few circuits that can be treated in such a simple manner; great care should be taken if any are so treated to make sure that the limitations of the near zone are met.

Before the radiated field is discussed a word should be said about that intermediate zone where both the induction field and the radiated field are present In the region where the distances involved are of the same order as the wavelength divided by 2π, the most intricate form of the equations has to be employed. This is especially true when two antennas are situated close to one another. The intermediate zone problems are very complex and for that reason many have not yet been solved.

5·5 THE RADIATION FIELD

The field in the region distant from the generator, often called the far region or far zone, is known as the radiated field. It is this field which contains the energy radiated from an antenna. For the far region, ρ is taken very large so that

$$\rho \gg \frac{\lambda}{2\pi} \tag{5·72}$$

Consequently,

$$\begin{aligned} \frac{i}{\rho} &\ll \frac{1}{v}\frac{\partial i}{\partial t} \\ \frac{q}{\rho^2} &\ll \frac{1}{v^2}\frac{\partial i}{\partial t} \end{aligned} \tag{5·73}$$

Examining the complete equations, Equations 5·69 and 5·70, we find that a large number of the terms can now be neglected. The equations reduce to only two components as follows, all other components being equal to or close to zero:

$$\begin{aligned} dH_\phi &= \frac{\sin\theta}{4\pi v\rho}\frac{\partial i}{\partial t}e^{-j\beta\rho}ds \\ dE_\theta &= \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{\sin\theta}{4\pi v\rho}\frac{\partial i}{\partial t}e^{-j\beta\rho}ds \end{aligned} \tag{5·74}$$

These two equations now express completely, for calculation purposes, the radiated field. This again does not mean that the other portions

of the equations do not exist but rather that they are so small that they may be completely neglected without any detrimental effect.

Notice that the ϕ component is at right angles to the θ component; hence the resultant vectors are always at right angles to one another. The magnitude of the **E** vector is equal to the magnitude of the **H** vector multiplied by the intrinsic impedance of free space, $\sqrt{\mu_0/\epsilon_0}$. For this reason only one integration is needed, and the other vector is obtained therefrom. Both vectors are in phase at every point in the far zone according to Equations 5·74. In this manner the spherical wave is similar to the plane electromagnetic wave. These results assume that there are no interfering or reflecting surfaces present in the zone.

Replacing the current i at the point s with its equivalent exponential form $I_{s(\max)}e^{j\omega t}$, where $I_{s(\max)}$ is the maximum value of the current in the small segment ds and ω is equal to 2π times the frequency of the current. Differentiating, we obtain

$$\begin{aligned} dH_\phi &= j\omega I_{s(\max)} \frac{\sin\theta}{4\pi v\rho} e^{j\omega t} e^{-j\beta\rho} ds \\ dE_\theta &= j\omega I_{s(\max)} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\sin\theta}{4\pi v\rho} e^{j\omega t} e^{-j\beta\rho} ds \end{aligned} \tag{5·75}$$

Taking the real part of the exponential, we find that the expression for dH_ϕ and dE_θ may be written in terms of the cosine of the phase relationships:

$$\begin{aligned} dH_\phi &= \left[\omega I_{s(\max)} \frac{\sin\theta}{4\pi v\rho} \cos\,(\omega t - \beta\rho + 90°)\right] ds \\ dE_\theta &= \left[\omega I_{s(\max)} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\sin\theta}{4\pi v\rho} \cos\,(\omega t - \beta\rho + 90°)\right] ds \end{aligned} \tag{5·76}$$

Examining Equations 5·76, we see again that at every point ρ distance from the generating source the electric and magnetic fields are in phase. Thus it correctly represents a spherical field inasmuch as the equiphase surfaces are spherical and travel radially outward from the origin. The field is polarized because only one component of each of the vectors is not zero. The amplitude, however, is not constant over a spherical surface but varies with the angle θ. The amplitude does not vary with the angle ϕ so that a circle determined by a constant ρ and constant θ will have both constant phase and constant amplitude. These circles can be pictured as the intersection of a z plane with any spherical surface about the origin.

5·6 Radiation pattern

A radiation pattern is a graphical representation of the magnitude of the electromagnetic vectors at a constant distance from the antenna. Examining Equations 5·76, we see that if ρ is considered to be constant, at any value, the shape of the pattern obtained will be the same as for any other value of ρ.

In Figure 5·4 is shown the polar pattern of a horizontal Hertzian doublet, which is an antenna so short that it can be considered to have a length ds and follow the pattern of Equations 5·76. In this case the

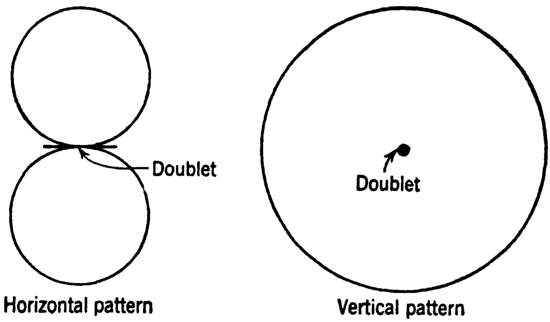


Fig. 5·4 The radiation patterns obtained from a Hertzian doublet which is horizontal to the reference plane.

z axis would be horizontal. Inasmuch as a three-dimensional pattern cannot be clearly shown on a flat sheet of paper, two separate patterns are presented. The horizontal pattern indicates the magnitude of the electromagnetic field vectors as the detector travels around the dipole in the horizontal plane at a constant distance from the antenna. In the pattern the length of line from the center to the line of the pattern is the relative magnitude of the vectors in that direction. We notice that the horizontal plane does not have to be the plane which includes the antenna but can be anywhere above or below the antenna, and the shape of the pattern will be the same. To avoid misunderstanding, the orientation of the antenna should always be shown on the pattern.

The vertical pattern is obtained in a similar manner with the detector going around the antenna at a constant distance from the antenna but in the vertical plane. Again, the vertical plane does not have to include the antenna in order to indicate the shape of the pattern.

In Figure 5·4 are shown the two patterns for the Hertzian doublet.

The horizontal pattern is an ordinary polar sine curve. It indicates that the maximum values in the field are obtainable in a direction perpendicular to the doublet and that the field is zero off the ends of the doublet. In a simple doublet like this, the intensity curve may be pictured as the apparent length of the doublet that an observer would see if he went around the doublet with the detector. When he is perpendicular to the doublet he sees a maximum length and when he is off the end of the doublet he sees zero length. The vertical pattern is circular since the orientation of the doublet with a detector is completely symmetrical in the vertical plane.

Let us try now to picture the three-dimensional pattern. It will be a three-dimensional surface where the length of line from the center to the surface indicates the relative magnitude in the direction of the line. Its shape will be a toroid with the inner radius reduced to zero. The doublet is in the center of the toroid, normal to the plane of the toroid.

There are an infinite number of possible antenna radiation patterns, each dependent on the construction and feeding of the antenna. The radiation pattern determines the direction for the maximum values of **E** and **H** and shows, therefore, the direction in which the maximum power is being propagated. For that reason the pattern is often referred to as the directive pattern of the antenna.

5·7 Current distribution in antennas

The radiation pattern of an antenna is obtained by integrating equations 5·76 along the length of the antenna, taking into account the phase and amplitude of the current in each segment ds of antenna and the position of the segment ds. Hence the current distribution along the length of the antenna has to be known or assumed before the calculation can be made.

The current distribution along a two-conductor parallel wire line is known to be sinusoidal; but when the two wires are separated and spread out so that an antenna is obtained, the distribution is not necessarily sinusoidal any more. A study of the distribution of current along a symmetrical center-driven antenna gives a good deal of insight into this problem. The center-driven antenna is a length of wire $2L$ units with a generator of very small size located in the center of the wire. This problem cannot be directly analyzed from ordinary circuit concepts as in the case of an ordinary transmission line. There the conductors are situated so close together that the forces acting on any element of either conductor is confined to those exerted by neighboring elements of the circuit; these forces are all near zone effects.

In a well-built transmission line the radiation is negligible and can be neglected. These assumptions, near zone forces and negligible radiation, cannot be made for an antenna; here the forces exerted by the fields are also determined by far and intermediate zone effects, and radiation, of course, does take place.

There are two methods of attacking this problem. One method of solution suggested by Schelkunoff and Feldman* takes into account the effect of radiation on the current and charge in an antenna by adding a simple term to the resistance of the wire. The voltage is assumed to be unaffected. In this manner transmission line theory may be applied to the problem and correct results obtained. Calculations based on the assumption that the antenna and the space surrounding it are two wave guides have also been made by Schelkunoff.† Another method of solution, first suggested by Hallén‡ and carefully studied by King and Harrison,§ is to analyze the antenna as a boundary value problem in electromagnetic theory. The solution for the current distribution is thus obtained and curves for the distribution of currents for a wide range of antenna lengths and ratios of length to radius are given in the literature.§ Although these curves are for the symmetrical center-driven antenna only, the concept can be carried over to other types of antenna configurations.

In Figure 5·5 is shown a center-driven symmetrical antenna commonly known as a dipole antenna. If the wire is very thin, in other words if the radius of the wire from which the antenna is constructed is vanishingly small, a sinusoidal distribution of current, which is conventionally assumed, will be a very good approximation of the true distribution. If L is the half length of the antenna and x the distance from the center to the point, S, under consideration, the current I_x at the point s is given by

$$I_x = I \sin\left[\frac{2\pi}{\lambda}(L - |x|)\right] \sin \omega^t \qquad (5{\cdot}77)$$

* S. A. Schelkunoff and C. B. Feldman, "On radiation from antennas," *Proceedings of the I. R. E.*, Vol. 30, pp. 511–516, November, 1942.

† S. A. Schelkunoff, *Electromagnetic Waves*, New York, D. Van Nostrand Co., 1943.

‡ E. Hallén, "Theoretical investigations into the transmitting and receiving qualities of antennas," *Nova Acta Royal Society of Science*, Vol. 11, pp. 1–44, November, 1938.

§ R. King and C. W. Harrison, "The distribution of current along a symmetrical center-driven antenna," *Proceedings of the I.R.E.*, Vol. 31, pp. 548–567, October, 1943.

where λ is the wavelength of the wave being radiated. This approximation is also fair for thicker antennas which do not exceed greatly a half wave in length. For the thicker, longer antenna the current varies from the sinusoidal distribution mentioned above. The current at the open end is still zero, of course, but the maximum value rises above the maximum in a sine wave distribution and the nodes

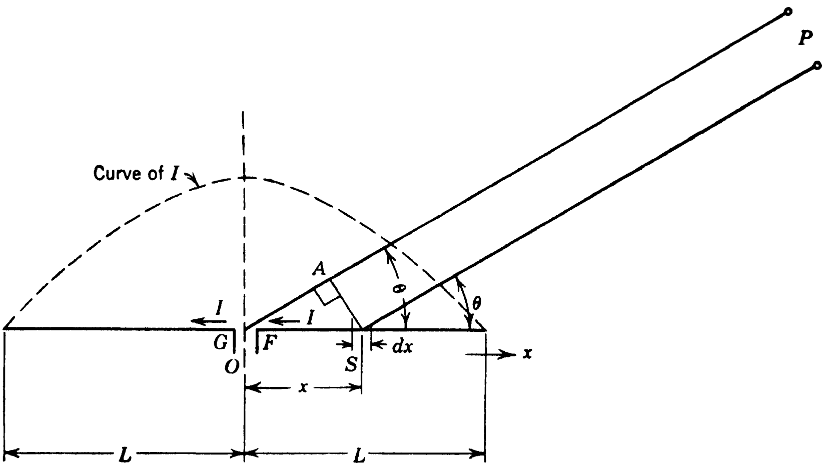


FIG. 5·5 A dipole antenna of length $2L$ units and a point P so far distant that lines connecting it with points on the antenna may be considered parallel

along the antenna, other than at the very end, do not come down to zero. For the most part, assuming a sinusoidal distribution of current in an ordinary antenna configuration will give a fair picture of the radiation pattern that will be obtained from the antenna when constructed.

5·8 The radiation pattern of a dipole antenna

Let us consider now the center-driven symmetrical dipole antenna shown in Figure 5·5. The wire is assumed to be very thin so that the sinusoidal current distribution, given in Equation 5·77, can be used. The wire of the dipole antenna, or any antenna for that matter, can be considered to be made up of a series of small Hertzian doublets placed end to end, as shown in Figure 5·3. The problem now is to obtain the vector sum of the radiations from all the Hertzian doublets at a distance ρ from a reference point on the antenna.

There are two important considerations to be taken into account:

One is the magnitudes and phases of the currents flowing in each of the doublets; these will determine the phase and the magnitude of the electromagnetic field vectors radiated from each of the doublets. The other consideration is the phase shift caused by different path lengths. The radiation from each of the doublets has to travel a different distance to the point of reception inasmuch as all the dipoles are at different distances from the antenna reference point; hence the sum of the radiations at the point of reception to be correct will also have to take into account the phase shift caused by this difference in path length.

In Figure 5·5 let us consider a small doublet of length ds, which in this case becomes dx, at the point x distance from the center of the antenna. We will now determine the phase and amplitude of the current at the point S on the line. In this case a sinusoidal distribution is assumed; hence, there is a standing wave along the line. Since it is open ended at x equal to L, the amplitude of the current there is zero and increases with the sine of the distance, in electrical degrees, from the end. In a standing wave on a dipole, as in an open-ended line, the phase of the current remains constant for 180° and then changes in sign. However, as shown in Equation 5·77, the amplitude of the current depends only on the absolute value of the distance x. Therefore, the current will change abruptly at x equal to zero, the point at which the generator is located. If x is taken positive in one direction and negative in the other, two equations will be necessary to express the current relationships along the line:

$$I_x = I \sin\left[\frac{2\pi}{\lambda}(L - x)\right] \sin \omega t \qquad x > 0$$

$$I_x = I \sin\left[\frac{2\pi}{\lambda}(L + x)\right] \sin \omega t \qquad x < 0 \qquad (5\cdot78)$$

(Include the minus sign when substituting values for $-x$.)

I_x is again the current at the point s and I is the maximum value of the current at the point $(L - x)$ equal to $\lambda/4$, provided the antenna is long enough. The $2\pi/\lambda$ factor converts the distance into radians. λ is the wavelength of the wave being propagated and ω is equal to $2\pi f$, where f is the frequency of the wave.

In obtaining the radiation pattern, all the constants in the equations for the electromagnetic field vectors, including ρ which is assumed constant for the pattern, can be lumped under a single constant K. This constant K can also include the time retardation $-\beta\rho$ and the 90° phase shift so that

$$K = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{1}{v\rho}\underline{/-\beta\rho + 90^\circ} \tag{5·79}$$

By substituting K into the equation for the electric field intensity from a doublet (in a plane parallel to the doublet) the second equation of Equation 5·76 becomes

$$dE_\theta = K\omega I_{s(\max)} \sin\theta \cos(\omega t + \phi)\, dx \tag{5·80}$$

The pattern for the magnetic field intensity will be the same; the only variation between the two will be a small change in the constant, as discussed previously.

The center of the antenna is located at the origin. To obtain the radiated field intensity at an angle θ and at a distance ρ from the origin, it is necessary to know the difference in phase between the received radiation from different points on the antenna. This difference in phase is noted as ϕ in Equation 5·80. It represents the difference in phase between the radiation coming from the origin and the radiation coming from the segment dx at the point x and is caused by the condition that they are at different points in space.

Let point P be the point at which the field intensity is desired. This point P is taken in the far zone so that the distance from the point P to the antenna is very much larger than the length, L, of the antenna. Hence the line connecting the point P and the origin and the line connecting the point P and the segment dx are so large, with respect to the distance x, that, for calculation purposes, at the antenna they can be considered to be parallel lines.

As shown in the diagram, dropping a perpendicular from the line Px to the line PO will determine the difference in length between the two distances. This difference in length in degrees is equal to ϕ. ϕ may also be expressed in radians by multiplying the distance OA with the factor $2\pi/\lambda$.

However, $I_{s(\max)}$ is also a function of x as shown in the expression for the current, I_x, at any point, x, in the antenna, Equation 5·78. Substituting $\left\{I \sin\left[\frac{2\pi}{\lambda}(L - x)\right]\right\}$ for $I_{s(\max)}$ in Equation 5·80 for one equation and using a plus x to obtain another equation, we find that the

two resultant expressions are

$$dE_\theta = K\omega I \left\{\sin\theta \sin\left[\frac{2\pi}{\lambda}(L - x)\right]\cos(\omega t + \phi)\right\} dx \qquad x > 0$$
$$dE_\theta = K\omega I \left\{\sin\theta \sin\left[\frac{2\pi}{\lambda}(L + x)\right]\cos(\omega t + \phi)\right\} dx \qquad x < 0 \tag{5·81}$$

To be able to integrate this expression, we have to express ϕ in terms of the variable x. From the diagram, shown in Figure 5·5, the distance OA is equal to $(x \cos\theta)$ so that ϕ in radians is given by

$$\phi = \frac{2\pi x}{\lambda}\cos\theta \tag{5·82}$$

Substituting Equation 5·82 into the two expressions for dE_θ, as given in Equation 5·81, we obtain the radiation at the angle θ of the segment of antenna dx:

$$dE_\theta = K\omega I \left\{\sin\theta \sin\left[\frac{2\pi}{\lambda}(L - x)\right]\cos\left(\omega t + \frac{2\pi x}{\lambda}\cos\theta\right)\right\} dx \quad x > 0$$
$$dE_\theta = K\omega I \left\{\sin\theta \sin\left[\frac{2\pi}{\lambda}(L + x)\right]\cos\left(\omega t + \frac{2\pi x}{\lambda}\cos\theta\right)\right\} dx \quad x < 0 \tag{5·83}$$

To obtain the radiation of the complete antenna in the direction of θ, it is required that Equation 5·83 be integrated over the complete length of the antenna. The limits used will be, for the first equation, from zero to L, and, for the second equation, from $-L$ to zero. Adding the two integrals, we obtain the full value of E_θ:

$$E_\theta = K\omega I \sin\theta \left\{\int_0^L \sin\left[\frac{2\pi}{\lambda}(L - x)\right]\cos\left(\omega t + \frac{2\pi x}{\lambda}\cos\theta\right) dx + \int_{-L}^0 \sin\left[\frac{2\pi}{\lambda}(L + x)\right]\cos\left(\omega t + \frac{2\pi x}{\lambda}\cos\theta\right) dx\right\} \tag{5·84}$$

To integrate Equation 5·84, the following integral equation * is used:

$$\int \sin(mx + a)\cos(nx + b) = -\frac{\cos(mx + nx + a + b)}{2(m + n)} - \frac{\cos(mx - nx + a - b)}{2(m - n)} \tag{5·85}$$

The quantities m, n, a, and b have the following values in Equation 5·84:

* B. O. Pierce, *A Short Table of Integrals*, New York, Ginn & Co., 1929, third revised edition, equation 479.

$$\begin{aligned}
m &= -\frac{2\pi}{\lambda} \text{ (for the first equation)} \\
m &= +\frac{2\pi}{\lambda} \text{ (for the second equation)} \\
n &= \frac{2\pi \cos\theta}{\lambda} \\
a &= \frac{2\pi L}{\lambda} \\
b &= \omega t
\end{aligned} \tag{5·86}$$

Using Equations 5·85 and 5·86, we find it possible now to write down the integrated equations for E_θ. Factoring out λ over 2π, which is a common factor in the denominator of the result, we get

$$\begin{aligned}
E_\theta = \frac{K\omega I\lambda \sin\theta}{2\pi} \Bigg\{ & \Bigg|_0^L \left[\frac{\cos\left[\frac{2\pi}{\lambda}(-x + x\cos\theta + L) + \omega t\right]}{2(1-\cos\theta)} \right. \\
& \left. + \frac{\cos\left[\frac{2\pi}{\lambda}(-x - x\cos\theta + L) - \omega t\right]}{2(1+\cos\theta)} \right] \\
& + \Bigg|_{-L}^{0} \left[-\frac{\cos\left[\frac{2\pi}{\lambda}(x + x\cos\theta + L) + \omega t\right]}{2(1+\cos\theta)} \right. \\
& \left. - \frac{\cos\left[\frac{2\pi}{\lambda}(x - x\cos\theta + L) - \omega t\right]}{2(1-\cos\theta)} \right] \Bigg\}
\end{aligned} \tag{5·87}$$

The term $\omega\lambda$ over 2π is equal to v, the velocity of propagation of the wave in free space. Putting in this simplification and substituting in the limits, as shown in Equation 5·87, we obtain

$$\begin{aligned}
E_\theta = KvI\sin\theta \Bigg\{ & \frac{\cos\left[\frac{2\pi}{\lambda} L\cos\theta + \omega t\right]}{2(1-\cos\theta)} \\
& + \frac{\cos\left[-\frac{2\pi}{\lambda} L\cos\theta - \omega t\right]}{2(1+\cos\theta)} - \frac{\cos\left[\frac{2\pi L}{\lambda} + \omega t\right]}{2(1-\cos\theta)}
\end{aligned} \tag{5·88}$$

$$- \frac{\cos\left[\frac{2\pi L}{\lambda} - \omega t\right]}{2(1 + \cos\theta)} - \frac{\cos\left[\frac{2\pi L}{\lambda} + \omega t\right]}{2(1 + \cos\theta)} - \frac{\cos\left[\frac{2\pi L}{\lambda} - \omega t\right]}{2(1 - \cos\theta)} \quad (5\text{·}88\ \textit{Continued})$$

$$+ \frac{\cos\left[-\frac{2\pi}{\lambda} L\cos\theta + \omega t\right]}{2(1 + \cos\theta)} + \frac{\cos\left[\frac{2\pi}{\lambda} L\cos\theta - \omega t\right]}{2(1 - \cos\theta)}\Bigg\}$$

Equation 5·88 can be used directly to obtain the radiation pattern of the antenna, but the equation can be better appreciated if the terms are combined and simplified. This is accomplished by multiplying the numerators and the denominators by the proper factor to make all the denominators equal to $1 - \cos^2\theta$, which in turn is equal to $\sin^2\theta$. The numerators are then combined using the equations for the cosines of the sum and differences of angles. Canceling the sine term in the factor in front of the brackets with one of the sine terms in the denominator, we obtain the equation for E_θ. This is, of course, the magnitude of the vector which has only a θ component. Putting in this equality, we obtain

$$|\mathbf{E}| = E_\theta = 2Kvl\left\{\frac{\cos\left[\frac{2\pi L}{\lambda}\cos\theta\right] - \cos\frac{2\pi L}{\lambda}}{\sin\theta}\right\}\cos\omega t \qquad (5\text{·}89)$$

Equation 5·89 gives the magnitude of the field intensity in any direction θ, as a function of θ, in a plane parallel to the dipole.

Since any one antenna can only have one length, the length L is a parameter which is the half length of the antenna in meters, provided λ is in meters. L always should be expressed in the same units as λ when substituting into Equation 5·89. Very often the total length of the antenna is given in wave lengths. Thus an antenna whose total length is said to be one wavelength, for instance, the dipole shown in Figure 5·5, has a ratio of L over λ equal to $\frac{1}{2}$. The advantage of using wavelengths as a measure of antenna length is that similar antennas having the same L over λ ratio have the same patterns. This result can be checked by examining Equation 5·89, the pattern for a dipole.

The pattern in a plane at right angles to the length of the dipole is obtained in a similar manner. However, inasmuch as the field intensity is independent of ϕ, the radiation pattern of a dipole in a plane at right angles to the length of the dipole will be a circle.

Example 5·1 Find the radiation patterns of a horizontal center-fed symmetrical dipole whose total length is equal to one half the wavelength of the wave being propagated.

The horizontal pattern is obtained by substituting into Equation 5·89. For a total length of $\frac{1}{2}\lambda$ the ratio of L over λ is equal to $\frac{1}{4}$. Substituting this value into Equation 5·89, we obtain

$$|\mathbf{E}| = 2KvI \left\{ \frac{\cos\left(\frac{\pi}{2}\cos\theta\right) - \cos\frac{\pi}{2}}{\sin\theta} \right\} \cos\omega t$$

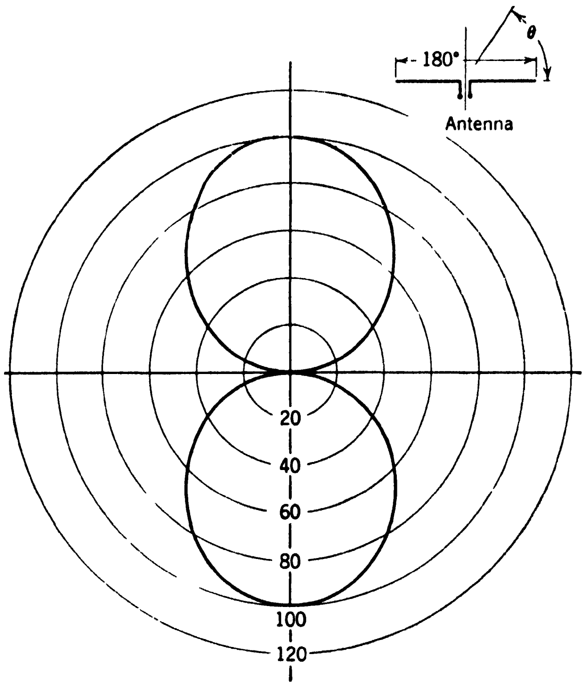


FIG. 5·6 The radiation pattern, taken in a horizontal plane, of a half-wave dipole antenna.

The cosine of $\pi/2$ is equal to zero so that the magnitude of $\mathbf{E}$ becomes

$$|\mathbf{E}| = 2KvI \left\{ \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right\} \cos\omega t$$

Only the term within the brackets varies with θ so that the horizontal pattern is obtained by plotting this quantity. The pattern is shown in Figure 5·6.

The vertical pattern of the dipole, neglecting the effect of the ground plane which normally is present, will be a circle. The effect of the ground plane will be studied in the next chapter. This concept of determining the

vertical pattern, neglecting the ground plane, although it is usually a fictitious condition, is very useful as will be indicated later.

EXAMPLE 5·2 Determine the horizontal radiation pattern of the antenna in example 5·1 when the operating frequency is doubled.

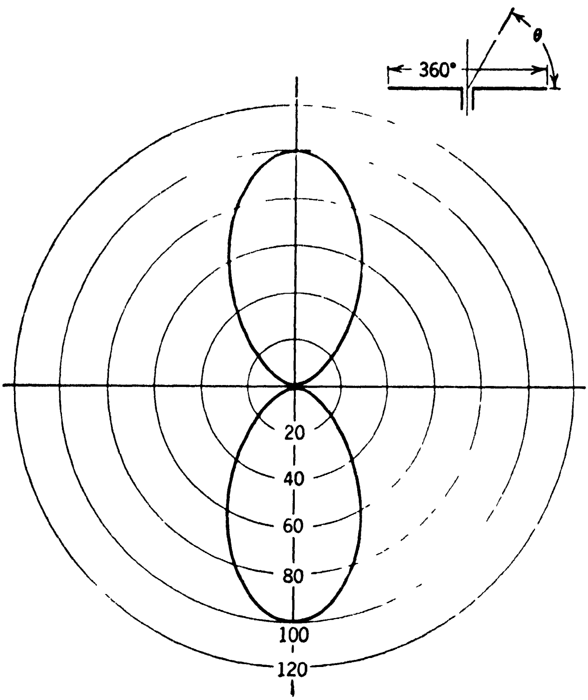


FIG. 5·7 The horizontal radiation pattern, taken in a horizontal plane, of a horizontal full wave dipole antenna.

If the operating frequency is doubled, the ratio of L over λ is doubled, becoming $\frac{1}{2}$. Substituting this value into Equation 5·89 and simplifying, we obtain

$$|\mathbf{E}| = 2Kvl \left\{\frac{\cos(\pi\cos\theta) + 1}{\sin\theta}\right\} \cos\omega t$$

Again, plotting the quantity within the brackets will give the desired pattern. This is done in Figure 5·7.

5·9 THE RADIATION PATTERNS OF A VERY SHORT DIPOLE

A very short dipole will be defined as a dipole whose L over λ ratio is small enough so that the sine of $2\pi L/\lambda$ can be taken equal to $2\pi L/\lambda$

itself. This is true to better than one per cent when L is one thirty-sixth of a wavelength or less. The pattern in the plane of the dipole will follow Equation 5·89. The form of the equation has to be modified to take advantage of the simplification stated above for a very short dipole. This modification is accomplished by substituting the product of two sine terms for the difference between the two cosine terms. Multiplying, also, the numerator and denominator by sin θ, we find that Equation 5·89 becomes

$$|\mathbf{E}| = 2KvI \sin\theta \left\{ \frac{-2 \sin\left[\frac{1}{2}\frac{2\pi L}{\lambda}(\cos\theta + 1)\right] \sin\left[\frac{1}{2}\frac{2\pi L}{\lambda}(\cos\theta - 1)\right]}{\sin^2\theta} \right\} \cos\omega t \tag{5·90}$$

The maximum value of the angles whose sines are taken in Equation 5·90 occurs when the quantities within the parenthesis is equal to two, one occurring when cosine θ is one and the other when cosine θ is minus one. This means that the maximum value of the angle will be $2\pi L/\lambda$. But for a very short dipole the sine of this angle is taken equal to the angle itself. Making this simplification and substituting the product of $(1 + \cos\theta)$ and $(1 - \cos\theta)$ for the $\sin^2\theta$ term, we obtain

$$|\mathbf{E}| = KvI\left(\frac{2\pi L}{\lambda}\right)^2 \sin\theta \left[-\frac{(\cos\theta + 1)(\cos\theta - 1)}{(1 + \cos\theta)(1 - \cos\theta)}\right] \cos\omega t \tag{5·91}$$

The quantity within the brackets reduces to one so that the final equation is

$$|\mathbf{E}| = KvI\left(\frac{2\pi L}{\lambda}\right)^2 \sin\theta \cos\omega t \tag{5·92}$$

Thus, similar to the pattern of a doublet, the pattern in the plane of the very short dipole is determined solely by sin θ. The pattern in a plane at right angles to the very short dipole will, as for the other dipoles, be a circle. The patterns, therefore, for a very short dipole will be the same as for a doublet. These patterns were illustrated in Figure 5·4. We notice that for a fixed current relationship along the dipole, the field intensity depends on the square of the length of the dipole and inversely as the square of the wavelength being used.

5·10 Radiation from a terminated wire in free space

In Figure 5·8 is shown a terminated wire in free space. The concept of a single terminated wire in free space is completely fictitious

but the problem is not. For instance, a terminated balanced transmission line may be analyzed by calculating the radiation pattern for each of the lines separately and then adding the results vectorially, taking into account the condition that they are displaced from one another in space. Since the radiation will be very small when the lines are electrically close together, the current distribution for this condition is assumed to be that of a terminated lossless transmission line.

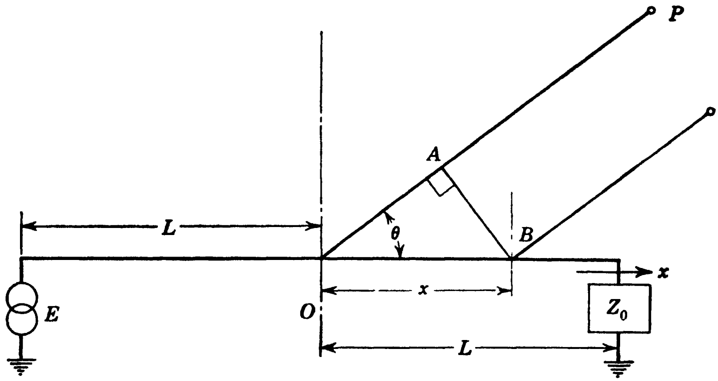


FIG. 5·8 A terminated wire in free space terminated in its characteristic impedance Z_0 with respect to a fictitious ground.

As the lines are spread apart the radiation is no longer very small, but a good approximation of the radiation pattern may still be obtained by making the same assumption for the current distribution.

A good example of the application of terminated wires in antennas is the rhombic antenna. It consists of a transmission line fed at one end and terminated at the other. The wires at the center of the line are spread apart to form a rhombus. In this case the transmission line constants vary along its length and there is attenuation caused by radiation. The current distribution is a very intricate function, but a fair approximation of the radiation pattern may be obtained by assuming a constant amplitude, varying phase, current distribution, similar to the current distribution in a terminated lossless line. Care must be taken in making these assumptions because an attempt to calculate impedances and radiation resistances using them will yield erroneous results.

Returning now to the wire shown in Figure 5·8, let us make the assumption that the distribution of current on it is of constant amplitude and varying phase, similar to the current distribution on a terminated transmission line wire without attenuation. The effect of the return

circuit for the current or of any ground plane is neglected and the wire is considered to be situated in free space.* The ground plane may be taken into consideration by means of images, as will be discussed in the next chapter.

The center of the antenna wire, as shown in Figure 5·8, will be used as the reference point for the radiation pattern. Thus the origin of the coordinate system is situated at the center of the antenna wire. The total length of the wire is $2L$, L being the distance from the origin to each end of the wire. The coordinate x is taken positive in the direction of the terminating impedance, and θ is measured from the line of the antenna wire.

The current at a point B, assumed to be x distance from the origin, now has to be determined. As in the case of a terminated lossless line, the phase shift along the line is obtained by multiplying the distance along the line by $2\pi/\lambda$. Using the point at the origin as a reference point for the phase of the current in the line, we find that the phase at the point B is $2\pi x/\lambda$ radians advanced from the current at the center. In other words, there is assumed to be a single traveling wave along the transmission line, having a constant magnitude, I, and traveling in the plus x direction from the generator to the load. The expression I_x for the current at any point x along the line is therefore given by

$$I_x = I \sin\left(\omega t - \frac{2\pi x}{\lambda}\right) \tag{5·93}$$

Differentiating I_x with respect to time in order to use it in the radiation equations, we obtain

$$\frac{\partial i_x}{\partial t} = \omega I \cos\left(\omega t - \frac{2\pi x}{\lambda}\right) \tag{5·94}$$

As shown in Figure 5·8, radiation from the point B and radiation from the origin O have to travel a different distance in order to arrive at the point P in the far zone. Again P is assumed to be so far distant with respect to the wire length that the line BP can be assumed parallel to the line OP. Since the radiation from B has to travel a smaller distance to the point P, it will actually introduce a leading angle, ψ, equal to the distance OA in degrees. Substituting Equation 5·94 into the equation for the dE_θ, Equation 5·76, which by the substitution of K

* A. Alford, "A discussion of methods employed in calculations of electromagnetic fields of radiating conductors," *Electrical Communications*, Vol. 15, p. 70, July, 1936.

may be simplified to the form of Equation 5·80, we obtain

$$dE_\theta = K\omega I \sin\theta \cos\left(\omega t - \frac{2\pi x}{\lambda} + \psi\right) dx \tag{5·95}$$

Since ψ is determined by the distance OA, it is given by

$$\psi = \frac{2\pi x}{\lambda} \cos\theta \tag{5·96}$$

Thus the quantity ψ is expressed in terms of the variable x. Equation 5·96 can now be substituted into Equation 5·95. E_θ is obtained by integrating Equation 5·95 between the limits of $-L$ and L:

$$E_\theta = K\omega I \sin\theta \int_{-L}^{+L} \cos\left(\omega t - \frac{2\pi x}{\lambda} + \frac{2\pi x}{\lambda}\cos\theta\right) dx \tag{5·97}$$

Performing the integration, we obtain

$$E_\theta = -\frac{K\omega\lambda}{2\pi} I \sin\theta \left| \frac{\sin\left[\omega t - \dfrac{2\pi x}{\lambda}(1 - \cos\theta)\right]}{\cos\theta - 1} \right|_{-L}^{+L} \tag{5·98}$$

Substituting in the limits of $-L$ and L, we get

$$E_\theta = 2KvI \left\{ \frac{\sin\theta \sin\left(\dfrac{2\pi L}{\lambda}\cos\theta - \dfrac{2\pi L}{\lambda}\right)}{\cos\theta - 1} \right\} \cos\omega t \tag{5·99}$$

Since the vector **E** has only one component, E_θ, Equation 5·99 determines the magnitude of **E** in any direction, θ, with the other space variables, ϕ and ρ, remaining constant. In other words, it determines the pattern in a plane parallel to the length of the wire. This plane is usually specified as the plane containing the terminated wire. Inasmuch as the quantity within the brackets is all that varies with the angle θ, a graph of that quantity will specify the radiation pattern given by Equation 5·99.

Again the radiation pattern in a plane at right angles to the wire will be a circle since the radiation is independent of the angle ϕ in this type of structure. We notice how this is obtained whenever all the doublets which are to be integrated are in a straight line in the antenna being analyzed.

Examining Equation 5·99, we see that the radiation pattern in the plane of the wire is not symmetrical around a line perpendicular to the antenna. This is checked by substituting θ plus 90° for θ and no-

ticing that the equation is no longer identical with Equation 5·99. Actually, most energy is transmitted in a line that makes an oblique angle to the direction along the wire towards the terminating impedance. This condition can be appreciated more clearly by assuming various values for L and substituting into Equation 5·99.

EXAMPLE 5·3 Determine the radiation pattern of a so-called terminated wire in free space which is a full wavelength long. Assume a current distribution along the wire which is constant in amplitude but varies in phase similar to the current in a lossless terminated transmission line.

The radiation from an antenna of this type is specified by the discussion in section 5·10. The pattern in the plane at right angles to the wire is again a circle. The pattern in a plane containing the wire is given by Equation 59·9. L in Equation 5·99 is equal, in this case, to $\lambda/2$. Substituting for L in Equation 5·99, we obtain

$$E_\theta = 2KrI \left\{ \frac{\sin\theta \sin(\pi \cos\theta - \pi)}{\cos\theta - 1} \right\} \cos\omega t$$

The quantity within the brackets determines the radiation pattern, which is plotted in Figure 5·9. Thus it is seen that the maximum of the first lobe tends to approach the direction of the wire. As the wire is made longer, the maximum of this first lobe will tend to approach this direction more and more closely.

5·11 FEEDING ANTENNAS

To radiate energy from an antenna, the energy has to be fed into the antenna in some manner, preferably an efficient transfer means from the source to the antenna. In some very special cases the antenna itself is a part of an oscillating circuit generating the r-f energy. In that case, the circuits are automatically matched, and the only special precautions to be taken are to see that no other parts of the circuit are radiating any waves which will interfere with the desired pattern and that no r-f energy is being lost through excessive resistance or currents in the circuit being used. Leakage to ground may also account for some losses.

In the majority of cases the antenna is located remotely from the generating source, and the energy is led to the antenna through a transmission line. This transmission line may be any of the types discussed in Chapter 1 or any special type that may be called for by the construction of the antenna. Feeding an antenna, like the dipole shown in Figure 5·5, by means of a balanced line is accomplished by considering the antenna to be a two-terminal impedance across the terminals G and F. One side of the balanced transmission line is con-

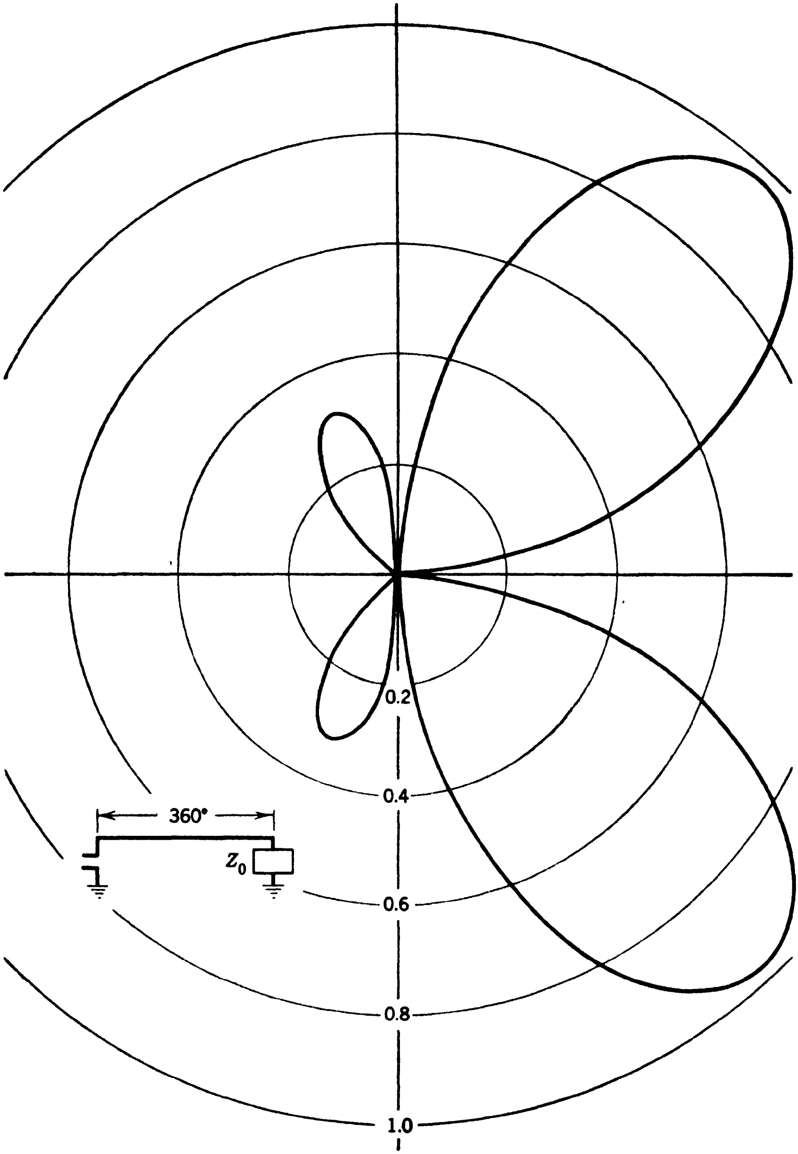


FIG. 5·9 The radiation pattern, taken in the plane containing the wire, of a terminated antenna, one wavelength long, in free space.

nected to G and the other to F. Since this is a balanced type of antenna no special precautions, other than checks for symmetry, have to be taken. If the antenna impedance is not the same as the characteristic impedance of the transmission line, matching means usually are employed. It can be of the stub type, discussed in Chapter 1, or any other type which will convert the antenna impedance to the characteristic impedance of the line.

Any balanced type of antenna, one which has two terminals that are balanced to ground, can be considered to be a two-terminal impedance, provided the two terminals are close enough together so that the normal

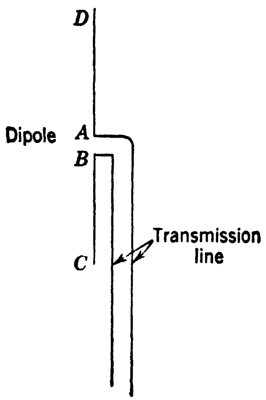


FIG. 5·10 A vertical dipole fed by means of a balanced transmission line. One side of the dipole is coupled to the transmission line because of parallel proximity.

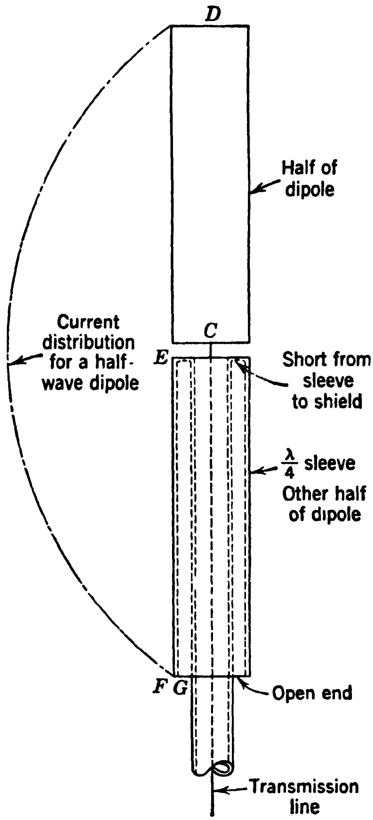


FIG. 5·11 A coaxial line feed for a vertical dipole antenna using a quarter wavelength sleeve.

transmission line equations can be applied to the transmission line connecting them to the source. In other words, neither wire of the transmission line should couple to any part of the radiating system.

In Figure 5·10 is shown a vertical dipole which is fed by means of a vertical transmission line. We notice the parallel proximity of the transmission line to one side of the dipole, namely side AC. This leads

to two basic defects: one, the impedance across the input is no longer balanced to the ground; and the other, some of the currents of the antenna will be introduced back into the transmission line. The second defect can be eliminated by shielding the transmission line but the shield will now have uncontrollable currents introduced on the outside of it.

However, this type of antenna adapts itself very well to a coaxial type of feed line. In Figure 5·11 is shown how this is accomplished. A sleeve one quarter wavelength long is placed over the shield of the coaxial line. It is shorted, at the end of the line, to the shield. It is open at the far end, the point F in the figure. The inside of the sleeve and the outside surface of the shield form a coaxial transmission line shorted at E. Thus the impedance looking in at FG is very high, forcing the current at the point F to practically zero. This current will now correspond very closely to the current at the open end of the other half of the dipole, CD. Plotting the current distribution for a half-wave total length now shows symmetry. Since the current on the inside of the shield is equal and opposite to the current on the coaxial center conductor, the center conductor is connected of one side of the dipole as shown. The shorting conductor for the sleeve conducts the current on the inside of the shield to the input of the other side of the dipole at E. To maintain completely symmetrical conditions, the half of the dipole projecting up into free space, part CD, is made the same dimensions as the sleeve.

Very often it is desirable to feed a balanced type of antenna with a coaxial line and vice versa. Special adapters to be used at the feed point can be constructed. The form that these adapters take and their analysis will be taken up in the chapter on complex transmission line network analysis, Chapter 8.

Where the losses in the transmission lines are negligible, because of low frequencies or short lengths, resonant lengths of transmission line or matching lengths may be used. For instance, if it is desirable to have the impedance of the antenna appear across the terminals of the generator, multiples of half wavelengths may be used. Quarter wavelength lines are often used to convert the impedance of the antenna to a more useful value.

Thus the antenna is considered to be the terminating load on a transmission line and, if the input connection is properly made, it can be treated like any load impedance, as discussed in Chapter 1. Its impedance can also be measured using the slotted line technique discussed in that chapter. Again, care must be taken that no radiated energy interferes with the measurements. It is best if the measuring equipment is completely shielded from the antenna field.

5·12 Radiation resistance

The input resistance of an antenna may be defined, for practical purposes, as the series resistance obtained by dividing the total power emanating from the antenna by the square of the rms current being fed into the input to the antenna. For comparative purposes between antennas, the current at the current maximum point in the antenna is used instead of the current at the feed point. Which is used is a matter of convenience, and one can be transformed into the other quite easily once the ratio of input current to maximum current is known. When the maximum current is used, the resistance obtained is called the radiation resistance.

Besides the input resistance caused by radiation, the antenna has a loss resistance resulting from the loss of power in causing the current to flow through the antenna wires proper. Often, the power absorbed by near-by objects and therefore not radiated, is included in the loss resistance. It is the ratio between the loss resistance at the input and the input resistance that determines the efficiency of the antenna. Thus, it follows, to get the highest ratio of radiated power to lost power, which would yield the highest efficiency, it is necessary to obtain the highest ratio of input resistance of the radiated power to loss resistance.

The radiation resistance is really a fictitious quantity which is useful in comparing different types of antennas but does not give a true picture of the operation of the antenna as a power source. Care must be taken to find out what resistances are referred to when reading the specifications of an antenna.

The total power being radiated can be obtained by taking the power passing outward through the surface of a sphere of arbitrary radius, ρ, surrounding the antenna. This is true because with waves traveling outward from the antenna, if none of the power is intercepted, the power passing outward through the sphere must be equal to the power being radiated in order to maintain equilibrium conditions; the conditions being that there cannot be an accumulation of undefined power within the sphere.

The power per unit area, ΔP, passing through the surface of the sphere, from Poynting's vector, is equal to the magnitude of the cross product of **E** and **H**. However, the vectors are at right angles to one another so that the cross product magnitude is equal to the product of their magnitudes. The subscript—θ, ϕ, ρ—denotes the point at which the vectors are taken:

$$\Delta P = E_{\theta,\phi,\rho} H_{\theta,\phi,\rho} \tag{5·100}$$

However from Equation 5·76, $H_{\theta,\phi,\rho}$ is equal to $E_{\theta,\phi,\rho}$ divided by the

intrinsic impedance of space, or, in other words, multiplied by $\sqrt{\epsilon_0/\mu_0}$. Substituting this factor into Equation 5·100, we obtain

$$\Delta P = \sqrt{\frac{\epsilon_0}{\mu_0}}\,(E_{\theta,\phi,\rho})^2 \tag{5·101}$$

Thus the power is expressed in terms of the field intensity only, where $E_{\theta,\phi,\rho}$ is the rms magnitude of the field intensity at the point θ, ϕ, ρ. To obtain the total power, P, passing through the sphere surface, Equation 5·101 has to be integrated over the whole surface. The limits of integration are taken as 0 to π for θ, and 0 to 2π for ϕ. Performing this step, we obtain

$$P = \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\,(E_{\theta,\phi,\rho})^2\,ds \tag{5·102}$$

where ds is a differential area of the surface. The area ds in terms of the variables is given by

$$ds = \rho^2 \sin\theta\,d\phi\,d\theta \tag{5·103}$$

Substituting Equation 5·103 into Equation 5·102, we find that the complete integral expression for power for a polarized spherical field becomes

$$P = \sqrt{\frac{\epsilon_0}{\mu_0}}\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}(E_{\theta,\phi,\rho})^2\rho^2\sin\theta\,d\theta\,d\phi \tag{5·104}$$

To obtain the expression for resistance, the power of Equation 5·104 has to be divided by the square of the rms current. Calling the resistance R, we obtain

$$R = \frac{\sqrt{\dfrac{\epsilon_0}{\mu_0}}}{I_{\mathrm{rms}}{}^2}\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}(E_{\theta,\phi,\rho})^2\rho^2\sin\theta\,d\theta\,d\phi \tag{5·105}$$

R is the input resistance if the current, I_{rms}, is the rms value of the current at the input to the antenna. It is the radiation resistance if the current I_{rms} is the rms value of the current at a current maximum point in the antenna. Actually the full name for input resistance should be input resistance of the radiated energy inasmuch as it does not include any of the loss resistance.

Sometimes the antennas are so short that the current standing wave does not come to its full maximum value. This occurs, for instance, in a dipole whose full length is less than a half wave. The current then used in the calculation of radiation resistance is that current which is

obtained if the current distribution curve is extended, following the distribution equation, until a maximum is obtained.

EXAMPLE 5·4 Obtain the radiation resistance and the input resistance of the half-wave dipole whose radiation pattern was obtained in example 5·1.

In example 5·1 it was shown that the field intensity is independent of ϕ and is given by

$$E_{\theta,\phi,\rho} = \frac{1}{2\pi\rho}\sqrt{\frac{\mu_0}{\epsilon_0}}\, I_{\text{rms}} \left\{ \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right\}$$

I_{rms} is used instead of I because the rms value of $E_{\theta,\phi,\rho}$ is desired. From the definition of I then, I_{rms} is the rms value of the current at the current maximum point.

An interesting feature of the half-wave dipole is that it is fed at a current maximum point. This means that I_{rms} is both the rms value of the current at the feed point and also at the current maximum point. Hence, in the case of a half-wave dipole, the radiation resistance and the input resistance will be the same. This is true of any antenna fed at a current maximum point.

Substituting the expression for the field intensity into the equation for R as given in Equation 5·105 and canceling out the two I_{rms} values which, in this case, are the same, we obtain

$$R = \sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{1}{4\pi^2}\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^2 \sin\theta\, d\phi\, d\theta$$

Simplifying by taking into account that the pattern is symmetrical for all eight quadrants of space, we obtain

$$R = \frac{2}{\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\int_{\phi=0}^{\phi=\pi/2}\int_{\theta=0}^{\theta=\pi/2}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^2 \sin\theta\, d\phi\, d\theta$$

Integrating first with respect to ϕ, we get

$$R = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\int_{\theta=0}^{\theta=\pi/2}\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta$$

This integral with respect to θ is rather difficult, and for illustrative purposes it will be accomplished graphically. The value of the constant is equal to the intrinsic impedance of space, 377, divided by π. This is equal to 120. Substituting this value into the expression for R, we obtain

$$R = \int_{\theta=0}^{\theta=\pi/2}\left[\frac{120\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right] d\theta$$

The function within the brackets is plotted in Figure 5·12. The area under the curve is divided into boxes, 4 ohms by 0.1 radian. Hence each box is equal to 0.4 ohm. The total number of boxes obtained, estimating those cut by the curve, is 182, and 182 times 0.4 yields 72.8 ohms as the resistance R. This is both the radiation resistance and the input resistance of the half-wave dipole.

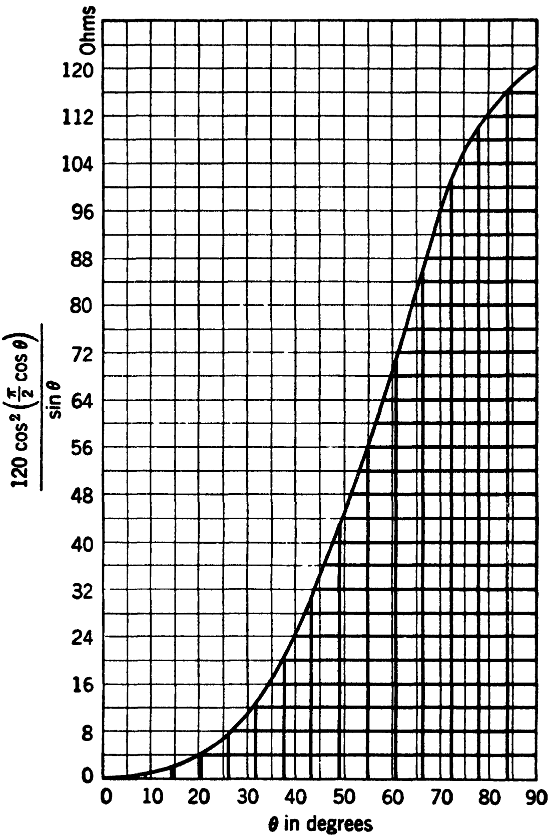


FIG. 5·12 Graphical integration of the equation for R in Example 5·4. Each box (outlined in heavy lines) is equal to 0.1×4.

When the equation is integrated mathematically, R comes out to be 73.1 ohms. The accuracy obtained by the graphical method depends upon the care with which the equation is plotted and the number of divisions used in the summation.

5·13 THE RECEIVING ANTENNA

The receiving antenna is another complex electromagnetic boundary value problem. The electromagnetic field impresses a voltage

along the length of the antenna giving rise to currents in the antenna proper. The problem is difficult to solve, but upon solution it resolves into an equivalent generator and an effective height. The equivalent circuit consists of replacing the antenna by a generator with an internal impedance equal to the input impedance of the antenna when used as a transmitting antenna. The internal voltage of the generator, which can be considered to be in series with the internal impedance of the generator, is equal to the effective height, h_e, of the antenna times the magnitude of the field intensity at the antenna. This is only true, however, when the antenna wire is in the equiphase plane of the electric intensity vector of the electromagnetic field.

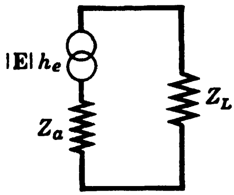


FIG. 5·13 The equivalent circuit of a receiving antenna of effective height h_e in a field with an electric intensity of **E**. Z_a is the input impedance of the antenna and Z_L is the load impedance.

In Figure 5·13 is shown the equivalent circuit of an antenna of effective height h_e, an input impedance of Z_a and a load impedance of Z_L. Calling E the magnitude of the field intensity, we find that the voltage across the load impedance, V, is given by

$$V = \frac{h_e E Z_L}{Z_a + Z_L} \tag{5·106}$$

The important factor to obtain, of course, is the effective height h_e. It depends on the configuration of the antenna and the current distribution therein. It has been solved in the literature* for some configurations of receiving antennas. For the case of a very thin, symmetrical, center-fed, dipole antenna the effective height, h_e, is given by

$$h_e = \frac{\lambda}{\pi} \tan\left(\frac{\pi L}{\lambda}\right) \cos\psi \tag{5·107}$$

where L is the half length of the antenna as shown in Figure 5·5 and λ is the wavelength being used. The antenna is assumed to lie in the equiphase plane of the received wave. The angle ψ is the angle between the antenna wire and the electric field intensity vector.

When the antenna does not lie in the equiphase plane the calculations become very complicated because the voltage being introduced

* R. King and C. W. Harrison, "The receiving antenna," *Proceedings of the I.R.E.*, Vol. 32, pp. 18–34, January, 1944.

C. W. Harrison and R. King, "The receiving antenna in a plane polarized field of arbitrary orientation," *Proceedings of the I.R.E.*, Vol. 32, pp. 35–49, January, 1944.

into the differential portions of the antenna are out of phase along the antenna. Thus Equation 5·107 cannot be used in those cases.

EXAMPLE 5·5 Determine the effective height of a very thin, symmetrical, center-fed dipole which has a total length of a half wavelength at 300 megacycles, the frequency at which it is being used. It is used parallel to the **E** vector.

At 300 megacycles the wavelength is one meter so that λ in Equation 5·107 is equal to one. Since it is used parallel to **E**, $\cos \psi$ is also equal to one. Substituting these values into Equation 5·107 with L equal to $\lambda/4$, we obtain

$$h_e = \frac{1}{\pi} \tan\left(\frac{\pi}{4}\right) = 0.318 \text{ meter} \qquad \textit{Ans.}$$

REFERENCE READING

G. W. PIERCE, *Electric Oscillations and Electric Waves*, New York, McGraw-Hill Book Co., 1920, Chapter 8 of Book II.

H. H. SKILLING, *Fundamentals of Electric Waves*, New York, John Wiley & Sons, 1942, Chapter XI.

J. A. STRATTON, *Electromagnetic Theory*, New York, McGraw-Hill Book Co., 1941, Chapters VII and VIII.

R. W. P. KING, H. R. MIMO, and A. H. WING, *Transmission Lines, Antennas, and Wave Guides*, New York, McGraw-Hill Book Co., 1945, Chapter II.

S. A. SCHELKUNOFF, *Electromagnetic Waves*, New York, D. Van Nostrand Co., 1943, pp. 331–342, 441–471.

PROBLEMS

5·1 Find the radiation patterns of a dipole antenna whose total length is 540°. Assume a sinusoidal current distribution.

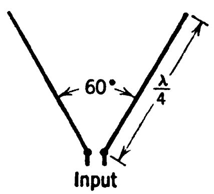


FIG. 5·14 V antenna of problem 5·2.

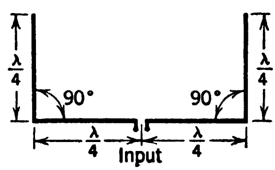


FIG. 5·15 U antenna of problem 5·4.

5·2 Find the horizontal radiation pattern of a horizontal V antenna fed at the apex in each if the legs of the V is 90 electrical degrees long and makes an angle of 60° between them. Assume a sinusoidal distribution of current (Figure 5·14).

5·3 Find the radiation pattern of the antenna in problem 5·2 when the angle of the V is increased to 120°.

5·4 Find the horizontal radiation pattern of the horizontal U antenna shown in Figure 5·15. Assume a sinusoidal distribution of current.

5·5 Find the radiation patterns of a terminated wire 270 electrical degrees long. Assume a constant-amplitude, varying-phase current distribution.

5·6 Find the horizontal radiation pattern of a horizontal V antenna fed at the apex if each of the legs of the V is one wavelength long and terminated at the end. The angle between the legs of the V is 90°. Assume a constant-amplitude, varying-phase current.

5·7 Find the radiation resistance of the antenna in problem 5·1.

5·8 Find the radiation resistance of the antenna in problem 5·3.

Chapter 6

ANTENNA ARRAYS

6·1 The purpose of antenna arrays

An antenna array consists of a number of radiators so arranged and having the proper current magnitudes and phases to obtain a specific pattern of radiation. It may make use of individually fed antennas, or parasitically fed antennas, or perhaps just a single antenna with a reflector surface, or any other possible combination. One use is to obtain a greater field intensity in the direction or directions desired by sacrificing radiation in other directions. As discussed in the previous chapter, individual antennas may be constructed to obtain various types of radiation patterns, but the number of different types that can be obtained is necessarily limited. With an array, by properly choosing the various elements, their position and their currents, practically any type of radiation pattern desired may be obtained.*

The shape of the pattern desired is determined by the use for which the antenna is being designed. In point-to-point communication, it is desirable to concentrate as much as possible of the power being fed into the transmitting antenna in the direction of the receiving antenna. In this manner, the highest field intensity possible with the amount of power available is set up at the receiving antenna. Very often it is a question of economy—whether a simple antenna array in conjunction with a transmitter with a high power output or a complicated antenna array in conjunction with a low power transmitter is cheaper. In point-to-point communication the actual shape of the pattern is not too important so long as the radiation in the direction of the receiving antenna is as large as desired.

For the receiving antenna, in point-to-point communication, it is desirable to have as large a voltage as possible set up in the load across the output of the receiving antenna when the signal is coming from the direction of the transmitting antenna and to have the sensitivity as low

* N. Marchand, "The design of antenna arrays by Fourier analysis," *Communications*, August, 1943; and I. Wolf, "Determination of the radiation system that will produce a specific directional characteristic," *Proceedings of the I.R.E.*, Vol. 25, p. 630, May, 1937.

as possible to signals coming from other directions. In this manner it is possible to avoid interference from extraneous signals which may set up comparatively large disturbances in the vicinity of the receiving antenna.

The desire for a specific horizontal pattern is fairly obvious, but in some cases a specific vertical pattern is also desirable. In short wave propagation the signal is transmitted along a straight line and does not follow the curvature of the earth. Normally this would mean that a receiving antenna beyond the horizon would not receive any signal. However, the ionosphere, which is an ionized region that exists in the upper atmosphere, reflects the electromagnetic waves at those frequencies back towards the earth. As an electromagnetic wave enters the ionized region, it sets up currents therein which absorb the incident wave and reradiate the energy back towards the earth. The 2-to-20 megacycle short wave frequencies, in use for transoceanic communication, have this property of reflection from the ionosphere. This now suggests a preferable vertical angle of transmission and reception inasmuch as reception in the short wave band beyond line of sight can be pictured as taking place through the medium of a giant mirror placed in the upper atmosphere. Vertical angle selectivity is often incorporated into point-to-point communication antennas for short wave communication.

In broadcast transmission it is desirable to concentrate the power in the vicinity of the most receivers; hence the densely populated regions should be covered in preference to the sparsely populated areas. The best location would be in the center of the residential portion of the city area. In that case an omnidirectional horizontal radiation pattern is usually employed. Often, though, conditions make it necessary to locate the transmitting antenna on the outskirts of the city or other area to be covered. Thus the antenna has to direct the transmitted signal over a wide area to one side of the antenna. It is usually accomplished by a simple array, but with the added difficulty that antennas for the broadcast frequency band are very large and even simple arrays are difficult to construct. Directivity in the vertical plane is used to concentrate the power in the horizontal direction for coverage of the adjacent areas.

Another use of antenna arrays, and perhaps one of the most important, is for search and navigational purposes such as radar, radio ranges, direction finders, radio beacons, aircraft blind landing systems, and ship guidance such as loran and shoran. The antenna arrays in most of these cases have to yield a radiation pattern meeting very close tolerances. For instance, for radar antennas it is often desirable to have

a pattern in three-dimensional coordinates which resembles a thin pencil. The minor lobes of the pattern have to be kept down to a negligible value In aircraft blind landing systems, the line that the plane is to follow must be accurately determined and no possibility of a false path can exist. It is usually accomplished by using the line of intersection of two accurately determined patterns. In all navigation, human lives are at stake and it is extremely important that no indications are false. Ideally, the system should be completely foolproof.

6·2 Directivity

The directivity of an antenna system,* sometimes referred to as the gain of the antenna, is a measure of the ability of the antenna to set up a certain field intensity in the direction desired with the power available. One method of determining the directivity of one radiating system compared to another is to determine the amount of power necessary to be fed to each one of them in order to set up the same field intensity at the desired point. Calling D_0 the directivity, P_0 the power fed into the standard radiating source, and P the power fed into the array or antenna whose directivity is desired; to set up the same field intensity with the antenna in question as the standard antenna with the input power of P_0, we find that

$$D_0 = \frac{P_0}{P} \tag{6·1}$$

The standard antenna is usually a fictitious antenna which has an omnidirectional pattern in any plane through the antenna. In other words, the three-dimensional space pattern would be a sphere. This appears to be a physically impossible antenna to obtain or construct but it makes a good standard to use Comparing a half-wave dipole to this standard antenna yields a directivity of 1 64 for the dipole in the direction perpendicular to the length of the dipole. Very often the half-wave dipole is used as a standard and, when used, the result will differ from the result obtained with the fictitious omnidirectional antenna by a factor of 1.64.

It is not necessary to use integration methods to obtain the powers used in the calculation of directivity if the radiation patterns and the input resistances are known. For instance, let us assume that an antenna array needs a current intensity at its input, I_a, which is n times the current intensity necessary in a half-wave dipole, I_d, to set up the

* G C Southworth, "Certain factors affecting the gain of directive antennas," *Proceedings of the I R E*, Vol 18, p. 1502, September, 1930.

same field intensity at the point desired. Both antennas are oriented so that the currents necessary are minimums. Calling the input resistance of the array at the point where I_a is measured, R_a, and of the dipole, R_d, the relative directivity, D, is given by the ratio of the powers fed into the antenna:

$$D = \frac{I_d^2 R_d}{I_a^2 R_a} \tag{6·2}$$

Substituting nI_d for I_a and canceling out the I_d^2, we obtain

$$D = \frac{1}{n^2}\frac{R_d}{R_a} \tag{6·3}$$

For the omnidirectional standard antenna the result would be 1.64 times that directivity obtained in Equation 6·3. Usually the loss resistances are left out in the calculation of directivity and only those resistances caused by radiation are used. When an array of more than one antenna is used the power fed into all of them has to be taken into account.

EXAMPLE 6·1 Determine the directivity, relative to a half-wave dipole, of an antenna which needs only one fourth the input current as a half-wave dipole to establish a similar field intensity at the point desired. The input resistance of the antenna is 90 ohms.

Substituting into Equation 6·3, where R_a is equal to 90, R_d from example 5·4 is 73.1, and n is ¼, we obtain

$$D = \frac{1}{(\frac{1}{4})^2}\frac{73.1}{90}$$

$$D = 13 \qquad \textit{Ans.}$$

6·3 MUTUAL IMPEDANCE BETWEEN ANTENNAS

When a receiving antenna is located in the far zone of a transmitting antenna, it can be considered to be coupled to the transmitting antenna inasmuch as a current in the transmitting antenna will cause a current to flow in the receiving antenna. However, in this case, where the receiving antenna is far distant from the transmitting source, its presence does not affect the input impedance of the transmitting antenna; the small current in the receiving antenna does not introduce another current back into the transmitting antenna. For this reason the receiving antenna is not considered to be coupled back to the transmitting antenna.

This is not true, however, when two antennas are placed close to-

gether as in an antenna array.* The current in one antenna will introduce a current in the other, causing the input impedances to vary. In multiple antenna arrays the calculations become very complex. By affecting the input impedances, these mutual couplings will change the voltages necessary to be impressed across the input of the antennas in order to obtain the desired currents in them.

Consider the general case shown in Figure 6·1 where two antennas are in close enough proximity so that they are coupled by a mutual impedance, Z_{12}. Each antenna is considered to be fed by a generator with an internal impedance: antenna 1 by V_1 with a series impedance Z_{L1} and antenna 2 by V_2 with a series impedance Z_{L2}. The input impedances of each of these antennas measured with the other antenna open-circuited will not necessarily be the same as the input impedance measured when the antenna is considered alone in free space because, even though the other antenna is open circuited at its input, currents will be introduced in the existing wires which affect the input impedance of the antenna being measured. Calling Z_1 the input impedance of antenna 1 in the open-circuited presence of antenna 2 and Z_2 the input impedance of antenna 2 in the open-circuited presence of antenna 1, we find that the equations for the voltages may be written down in terms of the current, I_1, in antenna 1 and the current, I_2, in antenna 2:

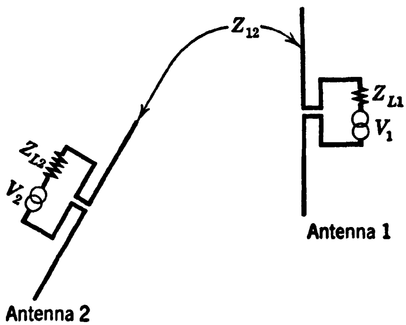


FIG. 6·1 Two antennas in close proximity so that they are coupled together by a mutual impedance Z_{12}.

$$\begin{aligned} V_1 &= I_1(Z_{L1} + Z_1) + I_2 Z_{12} \\ V_2 &= I_1 Z_{12} + I_2(Z_{L2} + Z_2) \end{aligned} \tag{6·4}$$

The currents I_1 and I_2 in these equations are the currents at the input terminals of the antennas.

Solving for the current I_1, we obtain

$$I_1 = \frac{V_1(Z_{L2} + Z_2) - V_2 Z_{12}}{(Z_{L1} + Z_1)(Z_{L2} + Z_2) - Z_{12}^2} \tag{6·5}$$

showing that the antenna input current is now dependent not only on

* P. S. Carter, "Circuit relations in radiating systems and applications to antenna problems," *Proceedings of the I.R.E.*, Vol. 20, p. 1004, June, 1932.

the voltage across the input but also on the voltage across the input of the other antenna and the impedances in the other antenna circuit. All these effects must be taken into account when designing an antenna array. These calculations are usually so complicated that in many cases the currents in the antennas in the final set up are adjusted by trial and error.

6·4 Two similar antennas fed in phase with equal currents

The calculation of radiation patterns for antenna arrays follows very closely the calculations made in the previous chapter for the individual antennas themselves. In those cases, the Hertzian doublet was considered an individual antenna and the result was obtained by summing up the contributions of all the doublets in the antenna by the process of integration. In an antenna array each antenna has its individual pattern, magnitude, and phase; and the resultant pattern is obtained by the vector addition in space of the contributions of the several antennas.

Let us consider the simple case of two similar antennas fed with equal currents in phase with one another. Since, because of symmetry, the input impedances of both the antennas will be the same, these relationships can be obtained by connecting the two antennas together with a length of transmission line and feeding the input power into the center of the connecting line.

Let us assume now that each antenna is a vertical dipole. The horizontal pattern of such an antenna, taken alone, will be a circle, as shown in Figure 6·2. The equation for that pattern is

$$|\mathbf{E}(\phi)| = E \tag{6·6}$$

where $\mathbf{E}$ is the vector field intensity and E is the constant magnitude of that intensity when both ρ and θ are kept constant at some arbitrary value. The horizontal pattern, for the plane at right angles to the lengths of the two dipoles and passing through their centers, will now be obtained for the case where the antennas are spaced a distance $2d$ apart between centers. Very often the separation is noted in electrical degrees. To convert the distance d to electrical degrees, $S°$, when the wavelength being propagated is λ, we use

$$S = \frac{360}{\lambda} d \tag{6·7}$$

The distance between centers is now said to be $2S°$.

In Figure 6·3 is shown a top view of the layout of the antennas. Antenna A and antenna B are symmetrically spaced from a midpoint

O. This midpoint O will be used as the reference point for the pattern to be obtained and is usually called the center of radiation. In Figure 6·3 the line ON, which is perpendicular to the line connecting the two antennas, is used as the reference line from which the angle ϕ is measured. Thus the angle to the line from O to the arbitrary point P in the far zone is taken as ϕ. Inasmuch as P is distant from the antenna, the lines AP, OP, and BP can be assumed parallel in the calculations.

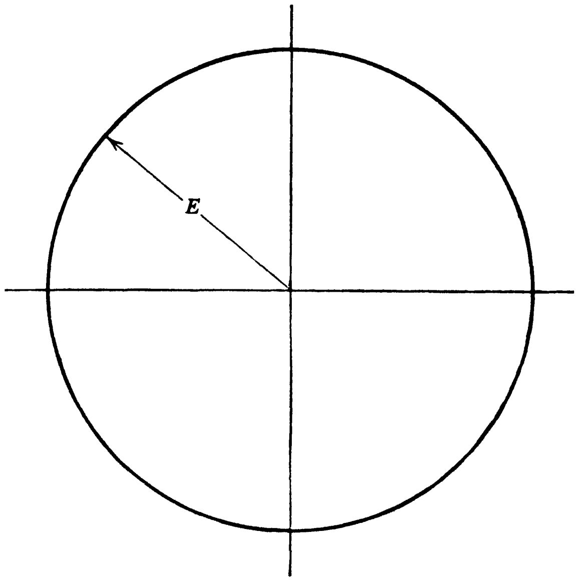


FIG. 6·2 The horizontal radiation pattern of a vertical dipole antenna.

The radiation from antenna A will have to travel a distance AP to reach the point P. Hence the field intensity at P set up by the radiation from A, to be referred to as E_A, is part of a traveling wave that has to travel a distance AC greater than the distance it would have had to travel had A been located at the point O. The point C was obtained by dropping a perpendicular from the point O to the line AP. The resultant phasor E_A, lagging $AC°$ (the angle obtained by multiplying the distance AC by 360° over λ), is shown in the phasor diagram in Figure 6·4.

Similarly the radiation from antenna B will reach the point P by a path which is OD shorter than the distance OP. The point D is again obtained by dropping a perpendicular from the point B to the line OP. The phasor E_B, which leads $OD°$ (the electrical degrees obtained by multiplying the distance OD by 360° over λ), and whose projection

represents the instantaneous magnitude of the field intensity set up at the point P by radiation from B, is also shown in the phasor diagram.

Inasmuch as the dipoles are parallel to one another, the two electric intensity vectors will have the same spatial direction. Hence the magnitude of the resultant field, E_T, may be obtained directly by adding the instantaneous magnitudes of the individual fields. Thus

$$E_T = E_A\underline{/-AC^\circ} + E_B\underline{/OD^\circ} \tag{6·8}$$

The actual differences in distance that the two antenna radiations have to travel, even though large with respect to a wavelength, are

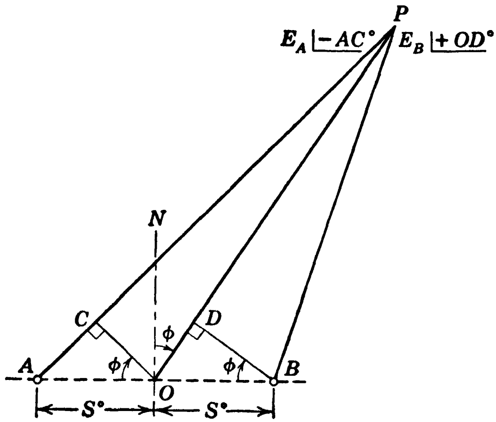


FIG. 6·3 Two antennas, A and B, spaced a distance $2S^\circ$ apart. The point P is so far distant from the antenna that lines connecting P to points on the antenna may be considered parallel.

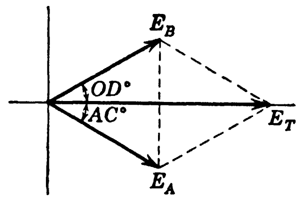


FIG. 6·4 Phase relationships at the point P of the radiation vectors from the two antennas, A and B, shown in Figure 6·3.

usually very small with respect to the distance OP. Thus the effects on the magnitudes of the individual radiations by these variations in distance are negligible. Inasmuch as the two antennas are similar and are fed with in-phase currents equal in magnitude, E_A will be equal to E_B. Calling them both E and noting from Figure 6·3 that AC is equal to OD, we obtain

$$E_T = 2E \cos AC^\circ \tag{6·9}$$

However, the angle AOC is equal to ϕ. Hence

$$AC^\circ = S^\circ \sin \phi \tag{6·10}$$

Substituting Equation 6·10 into Equation 6·9, we find that the equa-

tion for the magnitude of the resultant field intensity as a function of ϕ is given by

$$E_T = 2E \cos (S^\circ \sin \phi) \tag{6·11}$$

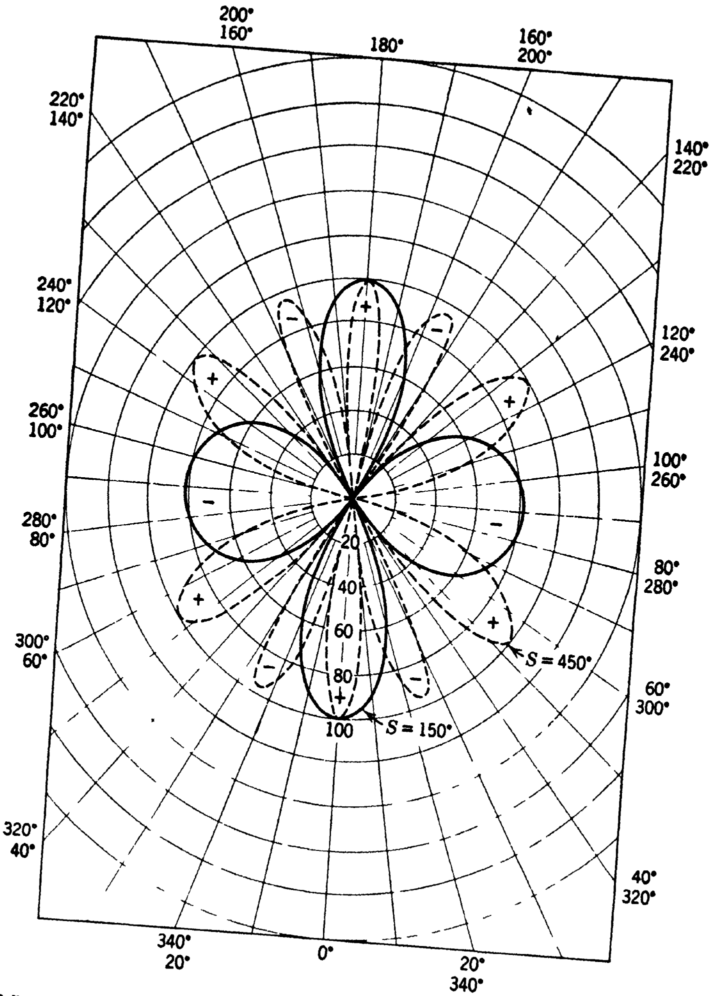
Equation 6·11 is the horizontal radiation pattern obtained by placing two antennas, which have omnidirectional similarly polarized horizontal patterns, a distance $2S$ electrical degrees apart and feeding them with the proper currents. If they are similar antennas, the currents will be equal and in phase. The value of E is obtained by calculating the magnitude of the field that would result at the point P if only one of the antennas (with the desired current to be used in the array flowing in it) were placed at the point O.

It is interesting to note, in Figure 6·4, that as the angle ϕ increases, the angular distances AC° and OD° increase equally so that the resultant, E_T, is always superimposed on the real axis in the diagram. Hence the phase of E_T, around the circular path for which the above radiation pattern was calculated, remains constant. In Figure 6·5 are shown two calculated patterns: one for a value of S° equal to 150° and the other for a value of S° equal to 450°. An interesting observation, upon examination of Equation 6·11, is that only two lobe patterns are obtained as long as S° is less than 90°. As the separation is increased, the number of lobes increases but does not disturb the result that a maximum always occurs along the normal to the line connecting the two antennas, at ϕ equal to zero.

When the individual antenna patterns are not omnidirectional in the plane in which the pattern is being calculated, the directional pattern for the individual antenna is substituted for E in Equation 6·11. This can be done because Equation 6·11 represents the sum of the individual radiations from each antenna. For instance, let us assume that the plane in which the radiation pattern is being calculated in Figure 6·3 was rotated 90° so that the two dipoles lay wholly in the plane parallel to the normal ON (a vertical pattern when using vertical dipoles). Calling the angle to the normal θ instead of ϕ, we find that the E in Equation 6·11 will be given by

$$E = E' \frac{\cos\left(\dfrac{2\pi L}{\lambda}\cos\theta\right) - \cos\dfrac{2\pi L}{\lambda}}{\sin\theta} \tag{6·12}$$

where L is the half length of the dipole. Calling the angle to the normal θ will change the angle ϕ to θ in Equation 6·11. Checking to see that both angles, the one in Equation 6·11 and the one in Equation



Courtesy of Communications

FIG. 6·5 The horizontal radiation patterns of two antennas having omnidirectional similarly polarized horizontal patterns, placed 2S° apart and fed with equal in-phase currents.

6·12, have the same value for the same direction, we can substitute Equation 6·12 into Equation 6·11:

$$E_T = E' \frac{\cos\left(\frac{2\pi L}{\lambda}\cos\theta\right) - \cos\frac{2\pi L}{\lambda}}{\sin\theta} \cos\,(S^\circ \sin\theta) \qquad (6\cdot 13)$$

Thus the field intensity E_T obtained in Equation 6·13 is equal to the field intensity obtained for antennas which have an omnidirectional pattern in the plane for which E_T is calculated times the individual pattern of the antennas in that plane. This assumes, of course, that all the antennas being used are similar. For this reason the variable factor, such as the cosine term in Equation 6·11, which is obtained using the omnidirectional individual patterns is often called an array factor. In other words, it is a factor which modifies the individual patterns when the individual antennas are a part of an array of similar antennas.

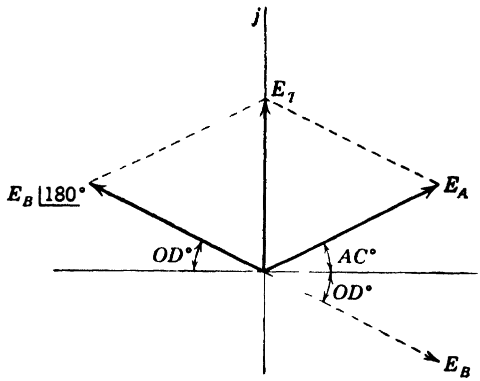


FIG. 6·6 Phase relationships of the magnitudes of the radiation vectors of the two antennas, A and B (as shown in Figure 6·3), when fed with equal currents 180° out of phase with one another

6·5 Two similar antennas fed 180° out of phase with equal currents

Let us assume now, in the antenna array shown in Figure 6·3, that the two antennas are fed with currents 180° out of phase with one another. Assuming the same type of omnidirectional pattern for antennas A and B, we can apply the same type of analysis. The phasor diagram will be similar to that of Figure 6·4 except that one of the phasors, say E_B, will be reversed. This difference will result in the phasor diagram shown in Figure 6·6. In this case the magnitude E_T will be given by

$$E_T = 2E \sin (S° \sin \phi) \tag{6·14}$$

and the phase of E_T is shifted 90° in reference to the radiation that would be received from antenna A if it were located at O.

As OD and AC increase with an increase in ϕ, the phasor E_T remains

at 90°; therefore, as a circle of constant radius is described around the center of radiation (the point O in Figure 6·4), the phase of the resultant field intensity remains constant. The horizontal pattern, which in this case is the pattern in the plane passing through the centers of the two vertical dipoles and at right angles to the dipole lengths, is now obtained by substituting the half spacing in degrees, $S°$, into Equation 6·14.

EXAMPLE 6·2 Determine the horizontal radiation pattern of two vertical dipole antennas in a horizontal plane which passes through the centers of the two dipoles. The frequency being used is 300 megacycles and the spacing between the centers of the antennas is 0.834 meter. They are fed 180° out of phase with one another

This problem meets the requirements of Equation 6·14. One item that has to be determined is the half spacing in electrical degrees, $S°$. The wavelength at 300 megacycles is one meter. Thus

$$2S° = \frac{360}{\lambda} 0.834 = 300°$$

or

$$S° = 150°$$

Substituting into Equation 6·14, we obtain

$$E_T = 2E \sin (150° \sin \phi)$$

The pattern is plotted in Figure 6·7.

EXAMPLE 6·3 What happens to the pattern of example 6·2 when the frequency of operation is doubled?

When the frequency is doubled the spacing in electrical degrees is doubled. This is caused by the condition that the wavelength is halved. It means that $S°$ of example 6·2 is increased to 300°. Substituting into Equation 6·14, we obtain

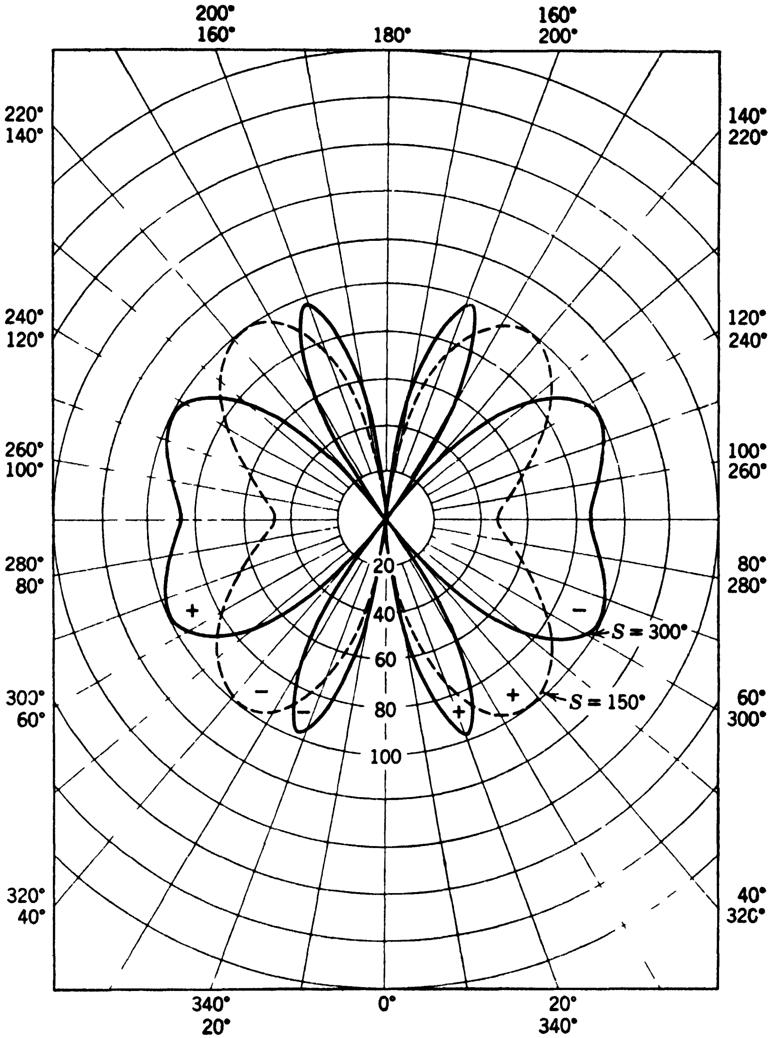
$$E_T = 2E \sin (300° \sin \phi)$$

This pattern is also plotted in Figure 6·7.

6·6 THE LOOP ANTENNA AS AN ARRAY

In Figure 6·8 is shown a simple square loop antenna with the loop in a vertical plane. If the length of wire in the loop, namely *abcdef*, is very much smaller than the wavelength being propagated, the instantaneous current flow in the wires throughout the loop can be assumed to be in phase. The antenna is fed across the input terminals a and f. The instantaneous current flow for one instant of time is shown in the diagram as I, with the accompanying arrows indicating direction. Inasmuch as this loop is assumed to have dimensions that are very small

with respect to a quarter wavelength, the currents in all the branches are shown equal and in phase.



Courtesy of *Communications*

Fig 6·7 The horizontal radiation patterns of two horizontally omnidirectional radiating antennas which are fed 180° out of phase with one another and spaced a distance 2S° apart.

Let us consider now the radiation pattern in the xy plane, the plane perpendicular to the plane of the loop and passing through the center

of the loop. There will be no horizontally polarized field (a field wherein the electric intensity vector is horizontal) in this plane because the horizontal current in eb is equal and opposite to the horizontal current in dc, and any point in the plane is equidistant from eb and dc. The only radiation from the loop which will set up an electromagnetic field in this plane is the radiation from the vertical elements bc and de.

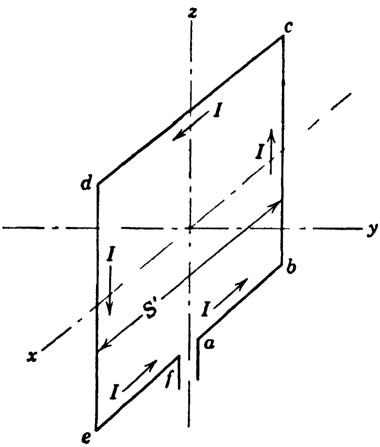


FIG. 6·8 A vertical loop antenna fed at the points f and a. The arrows indicate the current flow at one instant of time.

The radiation pattern of each of these wires, taken individually in this xy plane, will result in an omnidirectional pattern, namely, a circle. Since these wires are fed 180° out of phase, one current flowing upward when the other is flowing down, the combination of the two will yield a resultant radiation pattern that can be expressed by Equation 6·14. The intersection of the xy plane and the plane of the loop is used as the reference line, and the center of radiation is the center of the loop. However, it is well known that the sine of a small angle is equal to the angle itself expressed in radians (discussed in section 5·9). When S', the width of the loop in degrees, meets this requirement, the radiation pattern may be simplified. Substituting $S'/2$ for S in Equation 6·14 and taking the sine of the angle equal to the angle itself, we obtain

$$E_T = ES' \sin \phi \tag{6·15}$$

which is similar to the radiation pattern of a very short dipole where the dipole pattern is taken in the plane containing the dipole; however, the above pattern, Equation 6·15, is in the plane at right angles to the loop.

If the loop area is small enough, the pattern in any plane at right angles to the loop plane will be the same shape as determined by Equation 6·15 and shown plotted in Figure 6·9; it is also independent of the shape of the loop provided the shape is symmetrical. An equilateral triangle or a circle is an example of symmetrical shapes. The radiation pattern in the plane of the loop, with the center of the loop as the center of radiation, is nominally a circle. We notice how the pattern is again similar to a very short dipole when the radiation pattern of

the very short dipole is taken at right angles to the length of the dipole. The major difference between the patterns for the loop and the very short dipole is that the polarization of the wave is shifted 90°. In the circular pattern of the very short dipole the **E** vector is perpendicular to the plane in which the pattern is taken; whereas with the loop the **E** vector is in the plane of the pattern and at right angles to a line connecting the point at which it is observed to the center of the loop. In the loop the magnetic vector is at right angles to this plane; consequently, the loop antenna is sometimes referred to as a magnetic dipole.

A loop antenna often contains more than one turn of wire. It is often used at low frequencies where the loop size is extremely small

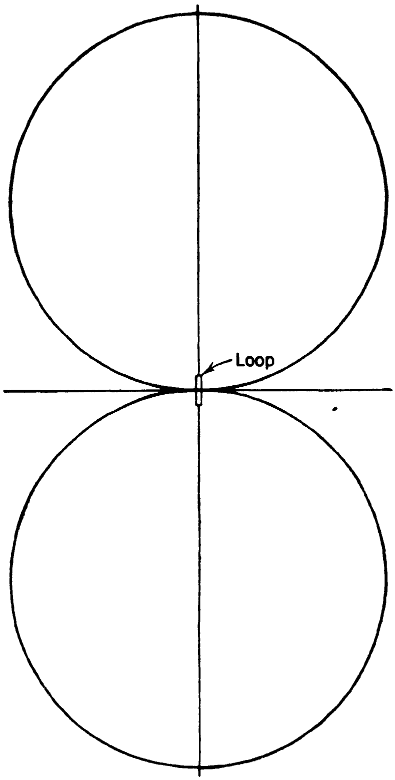


FIG. 6·9 The radiation pattern of a loop antenna in a plane at right angles to the plane of the loop

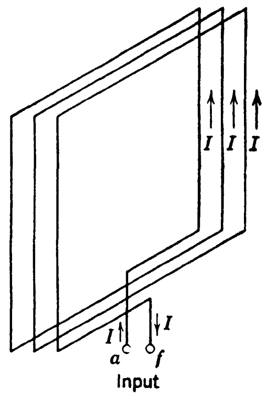


FIG 6·10 A three-turn square-loop antenna fed at the points a and f. The radiation resistance is increased by a factor of 9 over that of a single-turn loop.

with respect to a wavelength. A three-turn loop is shown in Figure 6·10. Again the loop is so small that the current can be considered to be equal and in phase in all the wires involved. We notice that now with the same current input at the input terminals the current is tripled in each side. The displacement of the wires in each side is so slight that it may be neglected and the radiation calculated as though the

total current were flowing through a single wire. Hence the field intensity is tripled. Tripling the field intensity with the same pattern and the same input current results in an increase in the input resistance of nine times.

In fact for an N turn loop the input resistance is increased by a factor of N^2. The input resistance of a small multiturn loop that is remote from ground* is

$$R = 31{,}200N^2 \frac{A^2}{\lambda^4} \tag{6·16}$$

where N is the number of turns, A the area of the loop, and λ the wavelength being used. A and λ should both be taken in the same units.

The field intensity in the plane at right angles to the plane of the loop at a distance ρ from the loop is obtained by substituting for E in Equation 6·15

$$E_T = \frac{120\pi^2}{\rho\lambda^2} NIA \sin \phi \tag{6·17}$$

where I is the current input to the loop and the other terms retain the same meaning as before.

In the ultrahigh frequency range of frequencies the loops are no longer very small with respect to a wavelength and the current can no longer be assumed constant. The distribution of current is usually assumed to be the same as for a shorted lossless transmission line. This distribution is a sinusoidal amplitude variation with constant phase along the length of the loop and with a maximum equidistant from the input. Only one turn is usually used. There are many variations striving to duplicate the low frequency loop and obtain constant current around the circumference of the loop.†

EXAMPLE 6·4 Determine the input resistance at one megacycle of a one-turn loop which encloses an area of 0.1 square meter. Assume that the loop is located remotely from ground.

The values given may be substituted directly into Equation 6·16 with λ equal to 300:

$$R = 31{,}200(1)^2 \frac{(0.1)^2}{(300)^4}$$

$$R = 3.85 \times 10^{-8} \text{ ohm} \qquad \textit{Ans.}$$

* A. Alford and A. G. Kandoian, "Ultra-high-frequency loop antennas," *Transactions of the American Institute of Electrical Engineers*, Vol. 59, p. 843, 1940.

† F. E. Terman, *Radio Engineers' Handbook*. Sec. 11, Par. 19, New York, McGraw-Hill Book Co., 1943.

Example 6·5 What is the effect of increasing the number of turns in example 6·4 to 50?

From Equation 6·16 we see that the input resistance increases as the square of the number of turns. Multiplying the resistance obtained in example 6·4 by $(50)^2$ or 2500, we obtain

$$R = 9.6 \times 10^{-5} \text{ ohm} \qquad \textit{Ans.}$$

6·7 The effect of the ground plane

When an antenna is in the vicinity of the ground, as it usually must be on all practical ground installations, any energy radiated towards the ground will be reflected and will modify the radiation pattern of the antenna in those planes in which it is effective and, hence, will affect the field intensities set up by the antenna. The field intensity may be calculated directly by using the reflection equations discussed in Chapter 4, but it is usually more convenient to use the method of images. In most practical antenna installations it is permissible to assume that the ground surface is a perfect plane reflector. The error introduced is usually negligible, but it will be pointed out later how the ground constants may be taken into account if necessary.

There are two important precepts in the reflection of the electromagnetic wave from a perfectly reflecting surface, in this case assumed to be a perfect conducting surface.

1. The electric field intensity vector parallel to the plane of the reflector is reflected with a 180° phase shift and equal amplitude, yielding the result that the tangential component of **E** is zero.
2. The electric field intensity vector normal to the reflecting surface is reflected equal in magnitude and phase to the incident vector so that the normal component of **E** is reinforced at the surface of the reflector.

Any electric field intensity vector can be considered to be made up of two components, one normal to the surface of reflection and the other tangent to it. The reflected wave is then calculated on the basis of the components and the result obtained by taking the sum of the reflected normal component and the reflected tangential component.

In Figure 6·11 is shown a dipole at A, a distance OA above a perfect ground plane located at O in the xy plane and parallel to the plane. The dipole is oriented parallel to the x axis. The radiation from A to the point P will consist of two rays, one direct from A to P and the other from A to the ground plane at R and reflected from R to P. At R, since in this case the electric intensity vector, **E**, is parallel to the reflecting plane, the **E** vector upon reflection will be reversed in phase.

According to the laws of reflection, the incident angle α will be equal to the reflection angle β.

Now, for calculation purposes, this reflected ray can be considered to come from the image A'. This image, A', is a mirror image of the antenna A. It is on a line drawn normal to the plane from A and is set back of the plane just as far from the plane as the antenna A is located in front of the plane. The distance $A'R$ will equal the distance AR.

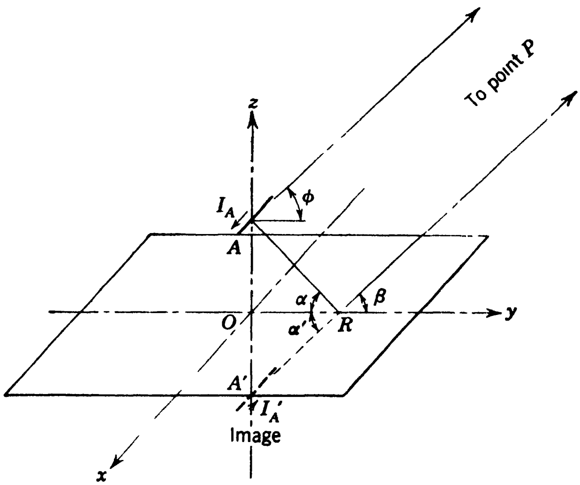


FIG. 6·11 Radiation from a dipole A parallel to the x axis and set a distance OA above a perfect ground plane at O.

Now let us assume that the plane no longer exists but that the image A' exists. To account for the change in phase that normally takes place at R, the current in A', I_A', is assumed to be the negative of the current in A, I_A. The direct ray from A to P is not affected but the reflected ray, from A to R to P, with its 180° phase shift, is now replaced by a ray from A' to P, going through R. Since AR is equal in length to $A'R$ and the 180° phase shift is taken care of by reversing the phase of the current in the image, the two cases are equivalent.

The vertical radiation pattern in the zy plane is now calculated by the method (discussed in section 6·5) used for two similar antennas fed 180° out of phase with equal currents. The angle between a line parallel to the y axis and a line in the direction of P is called ϕ. The pattern obtained in terms of the radiated field intensity magnitude E is

$$E_T = E\left\{2 \sin\left(\frac{2\pi}{\lambda} OA \sin\phi\right)\right\} \qquad 0 \leq \phi \leq 180° \qquad (6·18)$$

where λ is the wavelength being propagated. The angle ϕ is taken only between 0° and 180°, because in the case of a perfectly conducting plane the field intensity is everywhere zero below the plane. E is the magnitude of the field intensity when only the direct radiation from A was taken into account. The quantity within the braces in Equation 6·18 is a modifying factor for the radiation pattern of an antenna or an antenna array, when the electric field intensity vector being radiated is parallel to the ground plane and the radiating system is placed a distance OA above the ground.

If the electric field intensity vector being radiated is normal to the ground, such as may be obtained by substituting a vertical loop in the yz plane for the dipole at A, the voltage vectors will add at the ground plane. The radiation pattern for this case will be given by

$$E_T = E' \left\{ 2 \cos \left(\frac{2\pi}{\lambda} OA \sin \phi \right) \right\} \qquad 0 \leq \phi \leq 180° \qquad (6\text{·}19)$$

where E' is now the field intensity magnitude obtained by taking into account only the direct ray from the loop to P.

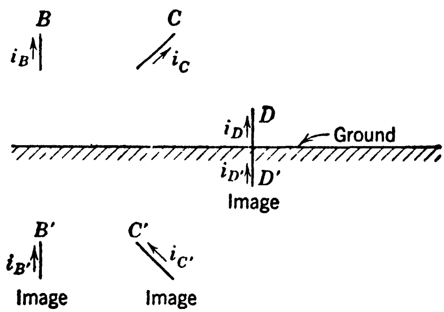


FIG. 6·12 Images of a vertical dipole, a 45° dipole, and a vertical monopole situated above a perfectly conducting reflecting plane.

We notice that the modifying factor in the case of an antenna or an antenna array which radiates an electric field intensity vector normal to the ground plane is now a cosine term. This is true because where the radiating elements are vertical, the current is assumed in the image to flow in the same direction as in the actual element. In Figure 6·12 are shown a number of differently oriented current elements and their images below a perfectly conducting ground plane.

An interesting case is the vertical monopole. This type of antenna is widely used in the broadcast frequency range. It consists of a vertical tower insulated from ground. It is usually fed by means of a

coaxial line, the inner conductor being connected to the base of the tower and the outer shield grounded at that point. In Figure 6·12 it can now be seen that the combination of the antenna and its image is exactly the same as a center-fed dipole where the height of the monopole is equal to the half length of the dipole. Hence the radiation

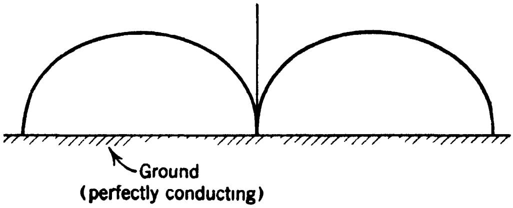


FIG. 6·13 The vertical radiation pattern of a quarter-wave vertical monopole.

pattern of the monopole will be the same as the dipole pattern that would be obtained if the dipole were substituted for the monopole and the ground plane. In Figure 6·13 is shown the radiation pattern of a quarter wavelength vertical monopole taken in the vertical plane. We notice how similar it is to one half the pattern obtained for a dipole a half wavelength long, shown in Figure 5·6. Because only one half the radiation of a dipole takes place in a monopole (half the space pattern is eliminated) the radiation resistance of a monopole is one half the radiation resistance of the equivalent dipole that would be obtained by assuming the image actually exists as a part of the antenna.

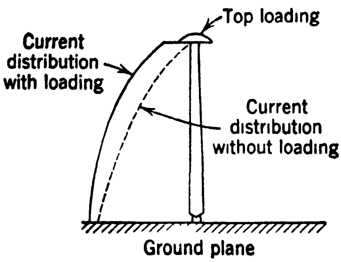


FIG. 6·14 The current distribution in a top-loaded vertical monopole illustrating the increase in current with loading.

Examining the pattern of the vertical monopole in Figure 6·13, we can see how well this antenna is adaptable to broadcast work. The energy is directed primarily to the horizontal plane so that it would cover the surrounding community. Very little is wasted by being propagated into the air in a vertical direction. The horizontal pattern, since it is that of a vertical dipole, is circular as it should be for broadcast work.

Sometimes, when a large antenna is too difficult to construct, a shorter vertical antenna with top loading is used. The top loading may consist merely of a large mass or equivalent structure which has a large capacitance to ground. This large capacity, from transmission

line theory, is equivalent to an added length of line when determining the current distribution. Hence the current at the top is no longer zero but is that which would be obtained at a point some distance before the end, depending on the loading. This means that the current all along the line will be increased, resulting in a more efficient radiator. The antenna is shown in Figure 6·14, where the current distribution without loading is shown as a dotted line and with loading as a solid line.

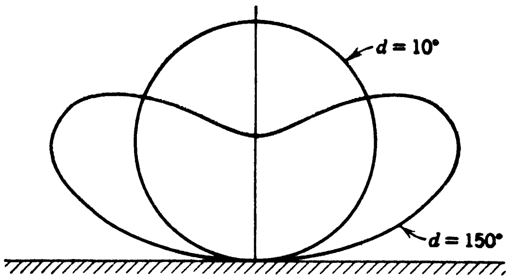


FIG. 6·15 The vertical radiation patterns of a horizontal dipole set a distance d degrees above a perfectly conducting ground plane.

EXAMPLE 6·6 Determine the vertical radiation patterns of a horizontal dipole antenna set 10° in electrical distance above the ground in the plane perpendicular to the dipole. Of one set 150° above the ground.

The pattern of the dipole in the plane perpendicular to it is a circle; hence all that is necessary is to plot the modifying factor as given in the bracketed term of Equation 6·18 with $\left(\frac{2\pi}{\lambda} 0A\right)$ equal to 10° and to 150°. This is done in Figure 6·15.

6·8 THE BROADSIDE ARRAY

The broadside array consists of similar antennas carrying similar currents and equally spaced along a straight line. The rectangular array consists of a number of similar broadside arrays stacked one above the other in a vertical plane, the distance between them being equal to the distance between antennas in the broadside array. It resembles the array obtained by putting a similar antenna at the intersection of every line of a checkerboard, where the number of vertical and horizontal lines need not be the same. These arrays are much used where a thin narrow beam is desired, such as in radar work.

In Figure 6·16 is shown a horizontal broadside array of horizontal dipoles, each oriented with its length along the length of the array.

Dipoles are often used in this type of array to cut down the side radiation. The dipoles are spaced $2d$ between centers and all have equal in-phase currents flowing in them. There are $2n$ antennas in the array, and the center of radiation is taken as the midpoint between the two center antennas.

The radiation in the plane passing through the line of the array, which in this case may be the horizontal pattern, can be obtained by adding together the radiation patterns of the n pairs of antennas. This

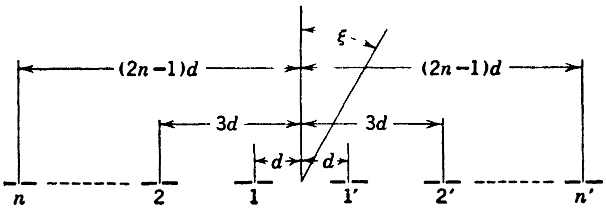


FIG. 6·16 A horizontal broadside array of horizontal dipoles spaced a distance $2d$ between centers.

can be done because each pair of antennas equally spaced from the center of radiation, the center of the array, will follow the analysis of two antennas fed in phase, as given in section 6·4, resulting, for any antenna pair, in an equation similar to Equation 6·11. The several field intensities obtained for each pair will be in time phase and, since they are originating from similar antennas, they will also have the same vector polarization.

Calling the magnitude of the field intensity set up by each dipole at a distance ρ and at an angle ξ to the normal of the array (that is, set up by each dipole if it were to be considered individually) E_ξ, we find that the sum of all the radiations is given by

$$E_T = 2\,E_\xi \cos\left(\frac{2\pi d}{\lambda}\sin\xi\right) + 2E_\xi \cos\left(3\,\frac{2\pi d}{\lambda}\sin\xi\right) + \cdots + 2E_\xi \cos\left([2n-1]\,\frac{2\pi d}{\lambda}\sin\xi\right) \quad (6\text{·}20)$$

The half spacing depends on d as shown in the diagram. E_ξ, however, is a function of the angle ξ also. First factoring it out of Equation 6·20, we obtain

$$E_T = E_\xi\left\{2\cos\left(\frac{2\pi d}{\lambda}\sin\xi\right) + 2\cos\left(3\,\frac{2\pi d}{\lambda}\sin\xi\right) + \cdots + 2\cos\left([2n-1]\,\frac{2\pi d}{\lambda}\sin\xi\right)\right\} \quad (6\text{·}21)$$

where the factor within the braces is the broadside array factor. Any type of antenna may be used if the individual antenna's pattern is substituted for E_ξ.

In this case of the dipole, E_ξ is given by Equation 5·89, where ξ is equal to θ plus 90°. It is equivalent to changing $\cos\theta$ to $\sin\xi$ and $\sin\theta$ to $\cos\xi$. Assuming a length of dipole equal to $2L$, we obtain for the equation for E_ξ

$$E_\xi = E\left[\frac{\cos\left(\frac{2\pi L}{\lambda}\sin\xi\right) - \cos\frac{2\pi L}{\lambda}}{\cos\xi}\right] \tag{6·22}$$

Combining Equation 6·21 and Equation 6·22, we obtain

$$E_T = 2E\left[\frac{\cos\left(\frac{2\pi L}{\lambda}\sin\xi\right) - \cos\frac{2\pi L}{\lambda}}{\cos\xi}\right]\underbrace{\left\{\cos\left(\frac{2\pi d}{\lambda}\sin\xi\right) + \cos\left(3\frac{2\pi d}{\lambda}\sin\xi\right) + \cdots + \cos\left([2n-1]\frac{2\pi d}{\lambda}\sin\xi\right)\right\}}_{\text{Array Factor}} \tag{6·23}$$

Equation 6·23 gives the radiation pattern of a broadside array of $2n$ horizontal dipoles fed with equal currents in phase. The dipoles are oriented parallel to the line of the array and the pattern is taken in a plane containing the line of the array. We notice the care that has to be taken to make sure that the angles referred to in radiation patterns to be combined always refer to the same axis. In many cases, such as above, some reference angles have to be changed to make all uniform.

The array factor shown in the braces of Equation 6·21 remains the same when the antennas are arranged in a vertical stack. This is known as a stacked or colinear array. The angle ξ in a stacked array is measured from a horizontal line, and the radiation pattern obtained is the pattern in a vertical plane. This type of array is used when it is desired to concentrate as much energy in the horizontal plane as possible at frequencies where the size of the array necessary is not too great. It is used extensively in the frequency-modulation band, where any energy directed skyward in a broadcast installation is not profitable. Stacked loops, with the plane of the loops horizontal, may be employed to obtain horizontally polarized radiation, or other antennas may be employed which yield the same result.

When a series of broadside arrays are stacked one above the other the array is known as a rectangular array. The spacing between the vertically stacked arrays is usually the same as the spacing between

the antennas in the broadside array, but it does not have to be. The array factor, shown within braces in Equation 6·23, will modify both the horizontal pattern and the vertical pattern of the individual antennas used in the array. If the array is made large enough, it will confine the radiated energy to a fine spatial pencil beam perpendicular to the plane of the array. Very often a reflector is used behind the array to concentrate the energy in one direction.

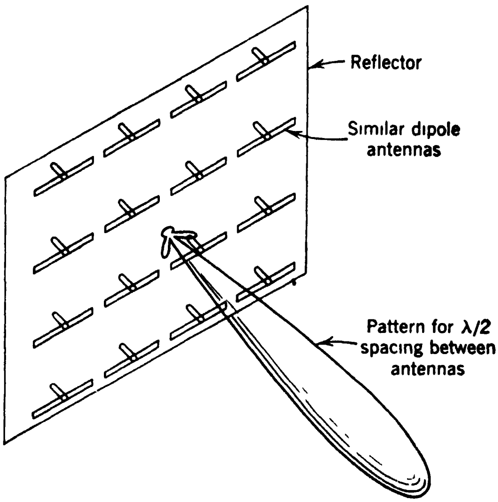


FIG. 6·17 Rectangular array with reflector showing the approximate spatial pattern for half-wavelength spacing between the centers of the antennas.

In Figure 6·17 is shown a 16-antenna rectangular array with a reflector. The approximate spatial pattern for half-wave dipoles spaced a half wave apart is shown. This pattern is obtained by substituting n equal to 2 and d equal to $\lambda/4$ in the array factor for a broadside array and then using it as a modifying factor for the vertical and horizontal pattern of a half-wave dipole. The factor is

$$\left\{\cos\left(\frac{\pi}{2}\sin\xi\right) + \cos\left(\frac{3\pi}{2}\sin\xi\right)\right\} \tag{6·24}$$

To cut down the side lobes, which are usually not desired, the current is sometimes tapered off towards the ends. That is to say, the currents in the end antennas are reduced from the normal currents flowing in the other antennas. This will be illustrated in the binomial array.

The problem of feeding power to a broadside array or a rectangular

array, at first glance, seems very difficult. They cannot be connected simply in parallel across the input voltage source because the mutual coupling between the antennas will cause the antennas in different parts of the array to have different input impedances. However, referring now to Equation 1·61 in Chapter 1, we find that the voltage, V_l, and current, I_l, at a distance l from the load end of a dissipationless transmission line is given by

$$V_l = E^+e^{j\beta l} + \rho E^+e^{-j\beta l}$$
$$I_l = I^+e^{j\beta l} - \rho I^+e^{-j\beta l} \tag{6·25}$$

At a distance l from the end of the line where $e^{j\beta l}$ is equal to j and $e^{-j\beta l}$ is equal to $-j$, which in the dissipationless line is a quarter wavelength, the voltage equation reduces to

$$V_{1/4} = jE^+ - j\rho E^+ \tag{6·26}$$

where $V_{1/4}$ is the input voltage to the quarter wavelength line. Substituting I^+Z_0 for E^+ in Equation 6·26, we obtain

$$V_{1/4} = jZ_0(I^+ - \rho I^+) \tag{6·27}$$

But the load current, I_L, flowing in the load impedance is given by the equation for the current in Equations 6·25, when l is zero; $e^{j\beta l}$ and $e^{-j\beta l}$ are both equal to one:

$$I_L = I^+ - \rho I^+ \tag{6·28}$$

Now substituting I_L into Equation 6·27 and solving for I_L, we get

$$I_L = -j\frac{V_{1/4}}{Z_0} \tag{6·29}$$

Equation 6·29 shows that for a quarter-wavelength line the current flowing into the load impedance, namely I_L, depends only on the characteristic impedance of the transmission line, Z_0, and the input voltage to the line. It is independent of the load impedance. This may be explained by the fact that the quarter-wavelength transmission line acts as a transformer to transform the load impedance to whatever value is necessary to obtain that result. It is true for any odd multiple quarter wavelengths except that care must be taken to prevent a reversal of phase. For a line equal in length to an odd number of half wavelengths plus a quarter wavelength the phase is reversed from that of a length equal to an even number of half wavelengths plus a quarter wavelength. Zero is considered to be an even number. The reversal

in phase may be taken into account by reversing the connections at the antenna.

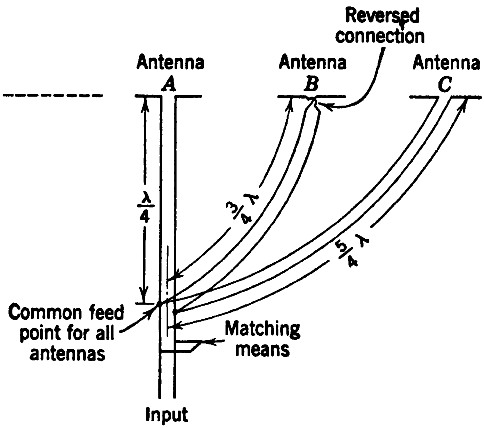


FIG. 6·18 A method of feeding a broadside array with equal in-phase currents by employing odd multiples of quarter-wavelength lines.

In Figure 6·18 is shown the connections to three antennas of a broadside array.* They are fed from a common point with transmission lines that are odd multiples of quarter wavelengths. We notice the reversed connection for antenna B because the transmission line in that case is an odd number of half wavelengths plus a quarter wavelength in length. A single feed point is not necessary for the whole array as long as each feed point used has the same input impedance as the others. For instance, the two halves of a broadside array when bisected at its center would be symmetrical so that the input can divide into two branch points. In this way the practical connections can be made easier. This type of branch feeding is illustrated in Figure 6·19.

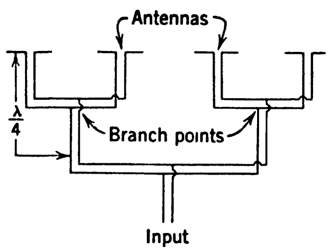


FIG. 6·19 Branch feeding of a broadside array where each antenna is located a quarter wavelength from a branch point.

EXAMPLE 6·7 Determine the horizontal radiation pattern of a horizontal broadside array consisting of 6 antennas spaced five-sixths of a wavelength apart. Each antenna is a half-wave vertical dipole and is fed with current at the same amplitude and phase as the others.

* John Ruze, *Design Considerations in Broadside Arrays*, Institute of Radio Engineers, Winter Convention, New York, 1946.

The horizontal radiation pattern of the individual antennas are circular so that E_ξ reduces to a constant E. Substituting 5/12 λ for d and 3 for n in Equation 6·21, we obtain

$$E_T = 2E\left\{\cos\left(\frac{5\pi}{6}\sin\xi\right) + \cos\left(\frac{15\pi}{6}\sin\xi\right) + \cos\left(\frac{25\pi}{6}\sin\xi\right)\right\}$$

This is shown plotted in Figure 6·20.

6·9 The binomial array

Let us consider a case concerning antennas which have omnidirectional horizontal radiation patterns like that of a vertical dipole. It is desired to combine a number of these antennas into an array which will have no minor lobes in its horizontal radiation pattern. By no minor lobes is meant that only two lobes should exist, one along the normal to the center of the array directed forward and the other along the normal directed to the rear. The rear lobe can always be removed by using the proper reflector back of the array.

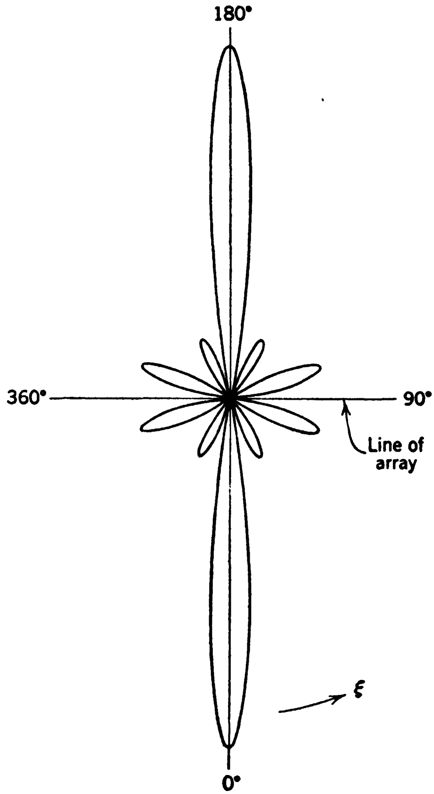


Fig. 6·20 The horizontal radiation pattern of a horizontal broadside array consisting of vertical dipoles spaced five-sixths of a wavelength apart.

The maximum directivity without minor lobes for a single pair of antennas is obtained when the antennas are spaced a half wavelength apart between centers. They are, of course, fed with equal currents in phase with one another. Since what is true for single antennas is also true for groups of antennas, two arrays can be spaced half a wavelength apart between centers of radiation for the same result. If neither of the arrays has minor lobes and their radiations are in phase, no minor lobes will be created. Thus two pairs of vertical dipoles are placed with their centers of radiation a half wave apart, as shown in Figure 6·21. The two pairs of antennas, one called A and B and the other C and D, are placed as shown so that B and C will coincide.

Since antennas B and C coincide, they may be replaced with a single antenna with twice the current flowing in it of either A or D. Thus only three antennas are used in a 1–2–1 arrangement, named for the current amplitude relationship that exists in the array. Two 1–2–1 arrays set a half wavelength apart will yield a four-antenna array with a current relationship of 1–3–3–1, the separation between antennas remaining at a half wavelength.

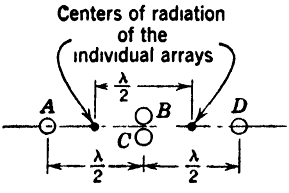


FIG. 6·21 Two pairs of antennas fed in phase and arranged to produce a radiation pattern with no minor lobes.

The method of obtaining the current relationships may be shown as a method of addition. For combining two pairs of antennas where the current relationships are 1–1, we use

$$\begin{array}{l} 1\text{–}1 \\ \quad 1\text{–}1 \\ \hline 1\text{–}2\text{–}1 \end{array}$$

and for combining two 1–2–1 arrays, we use

$$\begin{array}{l} 1\text{–}2\text{–}1 \\ \quad 1\text{–}2\text{–}1 \\ \hline 1\text{–}3\text{–}3\text{–}1 \end{array}$$

Continuing for the combination of two 1–3–3–1 arrays, we have

$$\begin{array}{l} 1\text{–}3\text{–}3\text{–}1 \\ \quad 1\text{–}3\text{–}3\text{–}1 \\ \hline 1\text{–}4\text{–}6\text{–}4\text{–}1 \end{array}$$

and so on until the number of antennas desired is obtained. Notice that for each addition the result is moved over one step and then added to itself. Actually this is a binomial expansion.

The horizontal radiation pattern for this type of array using vertical dipoles is given by

$$E_T = E\left\{\left[2\cos\left(\frac{\pi}{2}\sin\phi\right)\right]^{n-1}\right\} \tag{6·30}$$

where E is the magnitude of the field intensity obtained when an individual antenna, with a current flowing in it equal to the current in the end antenna of the array, is used alone at the center of radiation. The symbol n is the number of antennas in the final array.

Radiation patterns obtained for a 1–2–1 array and for a 1–3–3–1 array are shown in Figure 6·22. They are taken in the horizontal plane using vertical dipoles for the individual antennas. Again, the quantity

in the braces of Equation 6·30 is the array factor and can be used to modify the pattern of any antenna when that antenna is used in the array. The individual radiation pattern would be substituted for E in Equation 6·30.

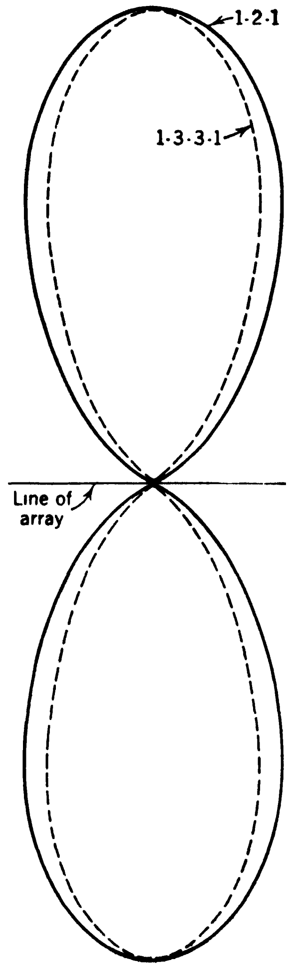


FIG. 6·22 Horizontal radiation patterns of horizontal 1–2–1 and 1–3–3–1 arrays consisting of vertical dipoles set a half wavelength apart.

6·10 THE END FIRE ARRAY

An end fire array consists of a number of antennas spaced equidistant along the line; in appearance it is very similar to a broadside array. However, in the end fire array, although the current amplitudes fed to the various antennas are the same, the phase relationships between currents in adjacent antennas are equal to the distance in electrical degrees separating the antennas. In this way the energy is directed in only one direction—along the line of the array—cutting down the rear radiation and concentrating it in a single forward direction.

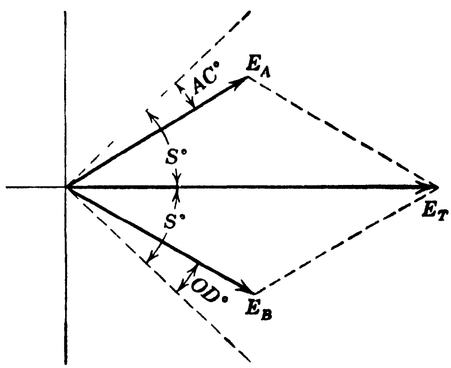


FIG. 6·23 A phasor diagram of the summation of the two field intensity vectors received at the point P from a two-antenna end fire array.

Let us consider two antennas spaced a distance of $2S°$ apart as shown in Figure 6·3 where antenna A is fed with a current that leads a reference current by a phase angle of $S°$ and antenna B is fed with a current

that lags the reference current by $S°$. Therefore, the radiated field intensity received at P from antenna A will lead the reference field intensity (obtained with antenna A at the origin with the reference current flowing in it) by an angle of $(S° - AC°)$ whereas the field intensity received at P from antenna B will lag by the angle $(S° - OD°)$. If vertical dipoles are employed in a horizontal array remote from ground, the horizontal radiation pattern of the array is obtained by using a circular radiation pattern for the individual antennas. Adding the two individual radiations, we obtain the phasor diagram of the summation of the two field intensities as shown in Figure 6·23. The resultant field intensity E_T is given by

$$E_T = E\underline{/S° - AC°} + E\underline{/-S° + OD°} \tag{6·31}$$

But $AC°$ is equal to $OD°$, which in turn is equal to $(S° \sin \phi)$. Making this substitution, we obtain

$$E_T = E\left\{2 \cos (S° - S° \sin \phi)\right\} \tag{6·32}$$

We notice from the phasor diagram that as $AC°$ and $OD°$ increase, the two phasors move closer and closer together. At the point where $AC°$ equals $S°$ the resultant field intensity will be a maximum. But $AC°$ is equal to $S°$ when ϕ is 90°, which means that the maximum will take place off the end B of the array. The factor in the braces is the array factor for a 2-antenna end fire array.

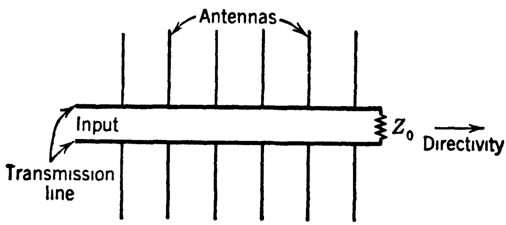


FIG. 6·24 A fishbone method of feeding an end fire array. The antennas form a part of a transmission line which is terminated in its characteristic impedance Z_0.

For the vertical pattern of the array, in a plane containing the line of the array, the factor in the braces still applies, but the vertical radiation pattern of the dipole has to be substituted for E. We notice how the end fire array concentrates the energy in both the horizontal and vertical plane. The same method of calculation may be applied to any number of antennas in an end fire array.

This type of antenna may be fed as shown in Figure 6·24. In this type of feed, called a fishbone antenna because of its resemblance to a fishbone, the antennas are hung on a transmission line terminated in

an impedance Z_0 which matches the line. The amplitude will have some attenuation along the line but, for purposes of calculating the radiation pattern, it may be neglected. The impedance Z_0 is not the characteristic impedance of the transmission line alone but is the value determined by the characteristics of the transmission line plus the loading put on by the antennas.

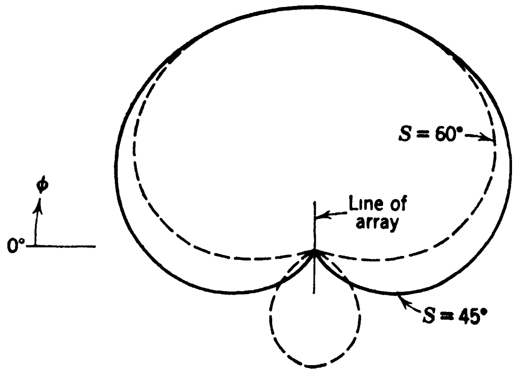


FIG. 6·25 The horizontal radiation patterns of a 2-antenna end fire array with a spacing between antennas of $2S°$.

EXAMPLE 6·8 Calculate the horizontal radiation patterns of a 2-antenna, vertical dipole, end fire array for a spacing between antennas of 90°; of 120°.

The patterns may be calculated by substituting directly into Equation 6·32, using first 45° for $S°$ and then 60°. The patterns are shown in Figure 6·25.

6·11 RHOMBIC ANTENNA

The rhombic antenna can be considered to be an array of four terminated wires (as used in Equation 5·99 of Chapter 5) for which a radiation pattern is illustrated in Figure 5·9. In Figure 6·26 is shown a horizontal rhombic antenna consisting of four terminated wires, their termination taking place in Z_0. Actually the termination can never be too good and the amplitude will vary along the line because of the loss through radiation. For calculation purposes for the radiation pattern, however, a constant amplitude, varying phase current can be assumed. Each wire makes an angle ψ to the horizontal, the angle being chosen so that the major lobes of the individual radiation patterns will coincide.

The wires are numbered from 1 to 4 as shown, with their centers of radiation labeled A, B, C, and D, respectively. The lengths of the wires are all equal to $2L$ as shown. Hence the currents in wires 2 and 3 when referred to the centers will lag a reference current by $2\pi L/\lambda$, and the currents in wires 1 and 4 will lead the reference current by an

angle of $2\pi L/\lambda$ radians. The horizontal radiation patterns of wires 1 and 3 now have to be shifted $-\psi$ degrees whereas the radiation patterns of wires 2 and 4 have to be shifted $+\psi$ degrees. This will take care of

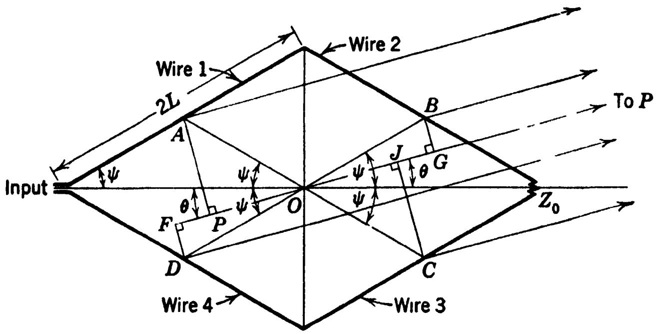


FIG. 6·26 A rhombic antenna considered to be an array of four terminated-wire antennas with their centers of radiation at A, B, C, and D.

the orientation of the antennas in obtaining the horizontal radiation patterns. We notice how closely this antenna resembles an end fire array.

Substituting these facts into the equation for the horizontal radiation patterns of the individual wires, Equation 5·99, and adding them, we obtain

$$\begin{aligned} E_T = {} & E\left\{\frac{\sin(\theta-\psi)\sin\left(\dfrac{2\pi L}{\lambda}\cos(\theta-\psi)-\dfrac{2\pi L}{\lambda}\right)}{\cos(\theta-\psi)-1}\right\}\angle\frac{2\pi L}{\lambda}-OP^\circ \\ & + E\left\{\frac{\sin(\theta+\psi)\sin\left(\dfrac{2\pi L}{\lambda}\cos(\theta+\psi)-\dfrac{2\pi L}{\lambda}\right)}{\cos(\theta+\psi)-1}\right\}\angle-\frac{2\pi L}{\lambda}+OG^\circ \\ & + E\left\{\frac{\sin(\theta-\psi)\sin\left(\dfrac{2\pi L}{\lambda}\cos(\theta-\psi)-\dfrac{2\pi L}{\lambda}\right)}{\cos(\theta-\psi)-1}\right\}\angle-\frac{2\pi L}{\lambda}+OJ^\circ \\ & + E\left\{\frac{\sin(\theta+\psi)\sin\left(\dfrac{2\pi L}{\lambda}\cos(\theta+\psi)-\dfrac{2\pi L}{\lambda}\right)}{\cos(\theta+\psi)-1}\right\}\angle\frac{2\pi L}{\lambda}-OF^\circ \end{aligned} \qquad (6·33)$$

The field intensity E is the magnitude that would be received at the, point P if each of the antennas was operating alone. $OP°$, $OG°$, $OJ°$ and $OF°$ are the angles obtained by multiplying the respective distances shown in Figure 6·26 by $2\pi/\lambda$. These distances were obtained by dropping perpendiculars from the centers of radiation to the line OP. However, AO, BO, CO, and DO are all equal to L. Hence

$$\begin{aligned} OP° &= OJ° = \frac{2\pi L}{\lambda}\cos(\psi + \theta) \\ OG° &= OF° = \frac{2\pi L}{\lambda}\cos(\psi - \theta) \end{aligned} \tag{6·34}$$

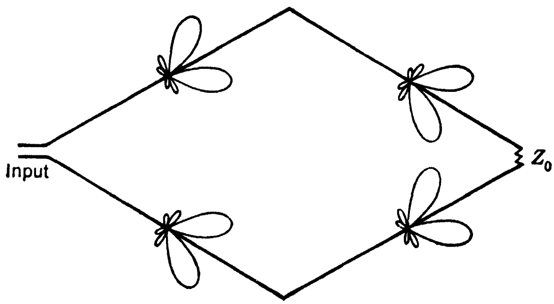


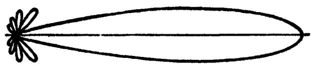
FIG. 6·27 A rhombic antenna showing the individual radiation patterns of each of its branches at their centers of radiation.

When Equations 6·34 are substituted into Equation 6·33 the horizontal radiation pattern for the rhombic is obtained. It is a complicated equation and does not allow of too much simplification. In determining the vertical radiation pattern we must take care because the vertical plane of the array does not include the line of the individual wires but makes an angle ψ to them. This type of an antenna also yields vertical directivity because all the lobes, which normally make an angle ψ to the vertical in a properly designed array, will add at that angle to the vertical.*

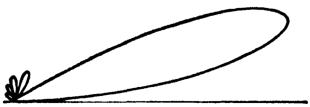
An understanding of what is taking place can be obtained by referring to Figure 6·27. The angle between the wires is so chosen that a major lobe of each wire points in the direction away from the input. They will add in an end fire fashion in that direction, giving a good directive result. Inasmuch as the antenna is terminated, the input impedance of the rhombic antenna will not vary too much with fre-

* E. Bruce, A. C. Beck, and L. R. Lowry, "Horizontal rhombic antennas," *Proceedings of the I.R.E.*, Vol. 23, p. 24, January, 1935.

quency. Also, if the antenna legs are made greater than two wavelengths long, the angle that the lobe makes with the wire changes slowly with frequency. This means that the rhombic antenna makes a good wide band antenna.



(a) Horizontal directive pattern.



(b) Vertical directive pattern, when the antenna is above a perfect ground plane.

FIG. 6·28 The shapes of the radiation patterns of a horizontal rhombic antenna with a leg length of approximately three wavelengths.

There is one fault with the rhombic antenna: about one half the power being fed into it, when it is being used as a transmitting antenna, is dissipated in the load resistor used to terminate the antenna. In spite of this, mainly because of its good directivity and its wide band feature, it is widely used. The loss in power is usually more than compensated for by the gain due to its directivity.

Figure 6·28 shows the shape of the horizontal and vertical radiation patterns for a horizontal rhombic antenna located above a perfect ground plane. We notice how its spatial pattern has a single major lobe extending upward at an angle. It is also used for receiving a down-coming wave that has been reflected from the ionosphere.

6·12 PARASITIC ANTENNAS

A parasitic antenna depends for its input power upon direct mutual coupling to a fed antenna. For instance, let us consider the two antennas shown mutually coupled in Figure 6·1 and suppose now that the source of voltage were removed from antenna 2. The input to antenna 2 is shorted to complete the circuit. Both V_2 and Z_{L2} go to zero in the equations for the currents in the antennas, Equation 6·4. The equations then reduce to

$$\begin{aligned} V_1 &= I_1(Z_{L1} + Z_1) + I_2 Z_{12} \\ 0 &= I_1 Z_{12} + I_2 Z_2 \end{aligned} \tag{6·35}$$

Using the second equation and solving for the current, I_2, in the second antenna, which in this case is said to be parasitically excited, we obtain

$$I_2 = -\frac{I_1 Z_{12}}{Z_2} \tag{6·36}$$

One of the first things that can be seen from Equation 6·36 is that, once the parasitic antenna is installed, the current flowing in it is directly proportional to the current in the excited antenna. Therefore, if the

current in the fed antenna is increased, in a finished installation, the radiation pattern will not vary because the current in the parasitic antenna will have increased a proportional amount.

The radiation patterns* are calculated like any other array pattern except that the currents in the parasitically excited antennas have to be calculated from Equation 6·36. Actually, the current in the parasitic antenna can be varied over very wide ranges of magnitude and phase angle by varying its tuning. Because the calculation of the mutual impedance is so very complicated, parasitic antennas are very often constructed on a trial and error basis. The antennas are usually constructed with the fed antenna stationary and the parasitic antennas movable so that their distances from the fed antenna are adjustable. The parasitic antenna tuning is also made adjustable, usually by making its length variable. The distances between the antennas and the tuning of the parasitic antennas are then adjusted until the best pattern of the type desired is obtained.

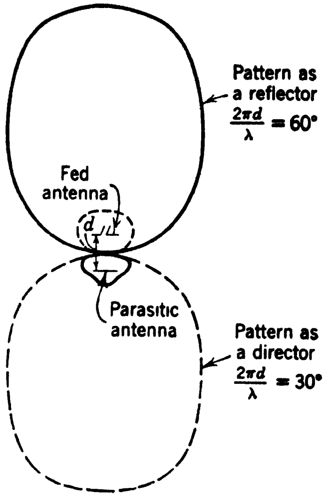


FIG. 6·29 A parasitic antenna used as a reflector and then as a director. Radiation patterns for both cases are shown when using nominal half-wavelength dipoles.

The commonest use for the parasitic antenna is as a reflector or director. This is really an end fire array where one or more of the antennas are fed by mutual impedance between the antennas. A reflector, in the case of a dipole antenna, is a conducting wire parallel to the dipole, spaced a small distance away from the dipole. It is shown in Figure 6·29, where a parasitic antenna is shown spaced a distance d away from the antenna.

For a reflector the distance is usually between an eighth and a quarter wavelength. To obtain the proper phase the impedance of the parasitic antenna has to be inductive. Hence, for a nominal half-wavelength reflector, tuned by varying the length, the actual length has to be greater than a half wavelength. A horizontal radiation pattern for a horizontal half-wave dipole and reflector, with the reflector spaced 60 electrical degrees away from the fed antenna, is shown in Figure 6·29.

* G. H. Brown, "Directional antennas," *Proceedings of the I.R.E.*, Vol. 25, p. 78, January, 1937.

As a director the parasitic antenna is placed in front of the dipole and the energy is directed forward. A director is usually placed less than an eighth of a wavelength in front of the antenna and its internal impedance has to be capacitive. Therefore, for a nominal half-wavelength director tuned by varying its length, the actual length is shorter than a half wavelength. The horizontal radiation pattern of a horizontal half-wave dipole and director, with the director spaced 30 electrical degrees away from the fed antenna, is shown dotted in Figure 6·29.

Sometimes both the reflector and the director are employed in an array to obtain the desired directivity. In some cases numbers of directors and reflectors are employed to obtain a large amount of directivity. In those cases the antennas are arranged in a straight line, being separated from one another by about 30 degrees of electrical wavelength. This is called a Yagi array. Besides being used back of and in front of the fed antenna, parasitic antennas are sometimes used to the side of the antenna to cut down the side radiation.

6·13 Parabolic reflectors

The use of a parabolic reflector to concentrate the light from a light source into a narrow beam is well known. The same principle is applied to a radio wave to obtain a narrow propagated beam. The parabola acts to convert the spherical wave originating at the antenna into a plane wave which comes out of the mouth of the parabola. For this purpose a metal reflector shaped into a parabolic surface is used. It usually measures at least several wavelengths across its mouth, the distance between the edges of the surface; hence the frequency for which it is to be used has to be quite high or the size becomes prohibitive.

In Figure 6·30 is shown a parabolic reflector with an antenna located at the focal point. The rays emanating from the antenna and being reflected by the parabola are shown as solid lines. We notice how, after leaving the parabola, the rays are all parallel. Since the distances traveled by all the rays from the antenna to the mouth are the same, they will also be in phase. This is, of course, what is desired to obtain a narrow concentrated beam. However, the direct rays from the antenna, shown dotted in the figure, will disperse in all directions over the area not covered by the reflector; the dispersion will depend on the radiation pattern of the individual antenna being used. This will cut down the directivity of the parabola.

To cut down the direct ray radiation, the antenna at the focal point

is made directive. Only a simple array is necessary, such as a director or reflector, to concentrate the radiation from the feed antenna into a unidirectional pattern toward the parabola; this will cut down the direct radiation. For very large parabolas, sometimes a small parabola is used on the feed antenna to direct the rays toward the large parabola.

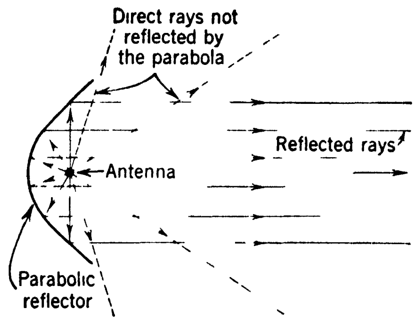


FIG 6·30 A parabolic reflector showing the direct and reflected rays from an antenna located at the focal point.

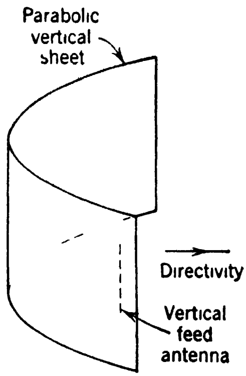


FIG 6·31 A parabolic reflector for a vertical antenna to yield a sharp horizontal pattern and a broad vertical pattern.

If it is desired to have very high directivity in one plane only, while leaving a fairly wide pattern in the plane perpendicular to it, a flat sheet bent in the form of a parabola is used, as shown in Figure 6·31. A vertical antenna is used and a vertical reflector sheet bent into a horizontal parabola acts as the reflector. The result is a sharp horizontal radiation pattern with a fairly wide vertical pattern. For directivity in both the horizontal and vertical planes a parabola of revolution is usually employed. The larger the parabola, the more concentrated the beam.

In the construction of parabolic reflectors the center of radiation of the exciting antenna is located at the focal point. For a cylindrical parabola with the electric vector parallel to the axis of the cylinder, as shown in Figure 6·31, the mouth plane, the plane determined by the edges of the front opening, can be located either in front of the feed antenna or back of it; there are no detrimental effects. However, where the electric vector is parallel to the parabolic curvature, as shown

in Figure 6·32, another effect is encountered. Calling a ray from the feed antenna in the direction of directivity of the parabola a *front ray* and using a dipole for the feed antenna, we encounter two types of reflections. The *front ray,* at one instant of time, will be polarized close to horizontal when it leaves the antenna and then will be vertically polarized, pointing up in the figure, when it is reflected. (Since the distances traveled by each ray is the same, the effect of the phase shift due to retardation effects will be the same for each ray so that it may be neglected in the discussion.) The *rear ray,* at that same instant of time, will also be vertically polarized when it is reflected but pointing down—not up as in the case of the front ray. Thus the *front* and *rear rays* will be in phase opposition and will cancel one another. Since this effect tends to cut down the forward radiation and therefore the directivity of the setup, it is not desirable. The way to eliminate this defect is to confine the size of the reflector being used to such a size that the plane of the mouth is never in front of the focal point, the point at which the feed antenna is located. The precaution need only be taken when the electric vector, or a component of it, is parallel to the plane of the parabolic curvature.

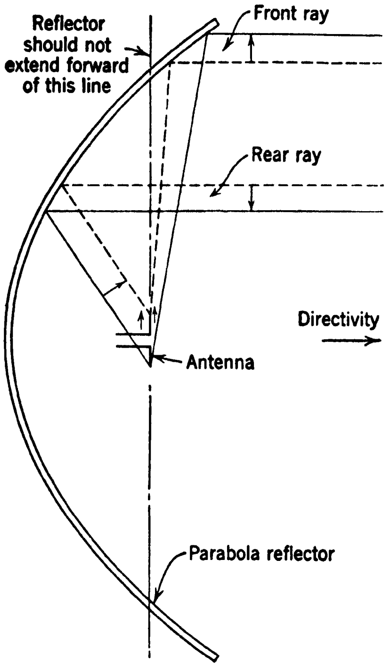


FIG. 6·32 A vertical parabolic reflector where the radiated electric vector is parallel to the plane of parabolic curvature.

6·14 RECEIVING ANTENNA ARRAYS

When an antenna, or an antenna array, is used as a receiving antenna, which means that it must abstract energy from a passing wave, its characteristics correspond in all respects to the same antenna, or antenna array, when used as a transmitting radiator. Even the parasitic antenna, which was excited by mutual impedance, now affects the receiving antenna in the same manner by coupling by mutual impedance into the receiving antenna the received energy. When plotted

for a receiving antenna, the radiation pattern represents the current flowing out of the output terminals for similar field intensities set up by waves arriving from different directions. This pattern will correspond to the radiation pattern of the antenna when used as a transmitting antenna. The input impedance of the antenna, now the internal impedance when used as a receiving antenna, will also correspond.

The reciprocal relationship between the transmitting antenna and the receiving antenna is expressed by the Rayleigh-Carson reciprocity theorem. This theorem states that if an electromotive force is inserted at the point A in one antenna and causes a current to flow at the point B in a second antenna, the same electromotive force inserted at the point B in the second antenna will cause the same current to flow at the point A in the first antenna. The only time that this theorem is not true is when the propagation is affected by an ionized medium in its transmission; under these conditions the path does not affect the wave in the same manner when its direction of travel is reversed. The theorem assumes that the same match is preserved for both the receiving and the transmitting antenna; otherwise, the transfer of energy between source, or detector, and the antenna will vary and the reciprocity theorem will no longer apply.

The reciprocity theorem is also very useful when taking antenna radiation patterns. It means that the antenna under consideration can be used as a transmitting antenna and the pattern taken with a receiver or it may be used as a receiving antenna and the pattern taken with a so-called target transmitter.

6·15 Taking radiation patterns

Throughout all the discussions it has been assumed that the polarization of the radiations, from the antennas being combined in an array, were all of the same type so that the magnitudes might be added directly, taking into account only the time phase difference involved. Actually, of course, when the polarizations are not the same the vectors have to be combined by vector spatial addition as well as phasor addition. The method of calculation is the same inasmuch as the field intensity vectors set up by each of the antennas at a generalized point P are obtained but the expressions for the field intensities are kept in their vector form and are added vectorially. As in the standard case, their time phases are referred to the time phase that would be obtained if one of the antennas were located individually at the chosen center of radiation.

Antenna patterns are obtained, practically, in several ways. In those cases where the antenna array is very large it is installed in a

fixed position above the ground with the ground plane parallel to the plane in which the pattern is desired. The ground should be flat and uniform with no reflecting objects in the environment to affect the pattern. It is not always possible, epecially at the lower frequencies, but for the best results the least disturbing position obtainable is usually chosen. Where the antenna is being used as a transmitting antenna, a portable receiver is carried around the transmitting antenna at a fixed distance from the antenna. Readings are taken every specified amount of degrees, depending on the accuracy desired. Care must be taken that the receiving antenna is always the same distance above the ground and that it is oriented the same way with respect to a radial line. This is done so that the sensitivity of the receiving antenna with respect to the transmitting antenna will not change at any point. The distance from the antenna under consideration has to be great enough

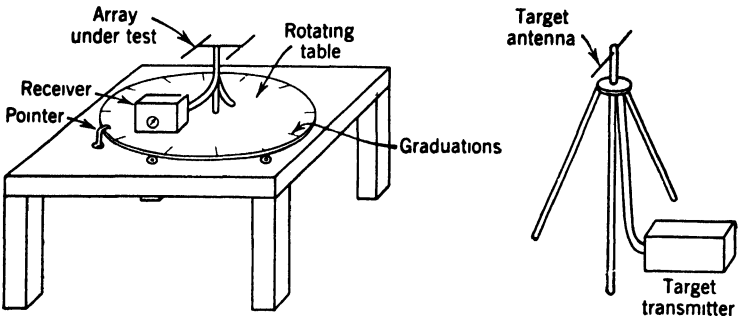


FIG. 6·33 Taking antenna patterns by means of a rotating table and a fixed target transmitter. The receiver and transmitter may be interchanged for convenience.

so that no near zone or intermediate zone effects are encountered. This means a distance of at least 5 and preferably 10 wavelengths to prevent erroneous results. Where the antenna under test is used as a receiving antenna, a small portable transmitter, called a target transmitter, is carried around the antenna. The same process is repeated and the same precautions have to be taken.

Where the frequency is high enough and enough patterns are to be taken, a circle of stakes is usually laid out, one every ten degrees or so, depending on the accuracy desired. The antenna under test is placed in the center and patterns taken by taking readings at each stake. At the very high frequencies it is not necessary to move the target antenna around in a circle. The antenna array is usually small enough so that it may be mounted on a rotating table, as shown in Figure 6·33. In this case the antenna under test is used as a receiving antenna and is

located on the rotating portion of the table. The plane of the table is the plane in which the radiation pattern is desired. A fixed target transmitter is used. The table is graduated, and readings are taken as the table is rotated. The orientation of the target transmitter and the distance from the array is kept very uniform with this method. The target transmitter antenna is sometimes made directive to avoid reradiation from objects in the environment.

This same system can be used with the antenna under test being a transmitting antenna and the target being a receiver. A great deal of care has to be taken in both cases to avoid interference with the pattern by either the person taking the pattern or the apparatus on the rotating table. They have to be located in a null of the pattern or far enough removed to avoid interference.

REFERENCE READING

F. E. Terman, *Radio Engineers' Handbook*, New York, McGraw-Hill Book Co., 1943, Section 11.

R. W. P. King, H. R. Mimo, and A. H. Wing, *Transmission Lines, Antennas, and Wave Guides*, New York, McGraw-Hill Book Co., 1945, pp. 121–156.

S. A. Schelkunoff, *Electromagnetic Waves*, New York, D. Van Nostrand Co., 1943, pp. 342–355.

PROBLEMS

6·1 Determine the horizontal radiation patterns of two parallel, horizontal, half-wave dipoles fed with equal in-phase currents separated by (*a*) three quarters of a wavelength, (*b*) one quarter of a wavelength, (*c*) one thirty-sixth of a wavelength.

6·2 Determine the horizontal radiation patterns of two parallel, vertical, half-wave dipoles fed 180° out of phase with equal amplitude currents and separated by (*a*) three quarters of a wavelength, (*b*) one quarter of a wavelength, (*c*) one thirty-sixth of a wavelength.

6·3 Determine the radiation patterns in a horizontal plane of four vertical monopoles placed in the corners of a square whose diagonals measure one eighteenth of a wavelength. They are so fed that the antennas connected by a diagonal get equal amplitude currents, but the antenna on one end of the diagonal is fed 180° out of phase with the one on the other end. (*a*) Find the resultant pattern when both pairs of antennas are fed equal currents. (*b*) Find the radiation pattern when one pair of antennas is fed double the current of the other pair. (*c*) Find the radiation pattern when they are fed with equal amplitudes but when one pair is fed 90° out of time phase with the other pair. (This problem can be worked out by obtaining the radiation patterns of each pair first and then combining the result, taking into account the feed currents.)

6·4 Determine the horizontal radiation pattern of a vertical loop antenna which is a single-turn square, one-eighteenth of a wavelength on each side. It is fed from the midpoint of a horizontal side.

6·5 Repeat problem 6·4 for a horizontal loop antenna of the same size fed from the midpoint of one side.

6·6 Repeat problem 6·4 for a square loop that measures four-ninths of a wavelength on each side.

6·7 Determine the horizontal radiation pattern of three quarter-wave monopoles fed with equal, in-phase currents, each monopole being located in the corner of an equilateral triangle, each side of which is half a wavelength long.

6·8 Determine the horizontal radiation pattern of a horizontal broadside array consisting of half-wave dipoles oriented so that their lengths extend along the length of the array and spaced 180° between centers, using (*a*) 4 antennas, (*b*) 8 antennas, (*c*) 5 antennas.

6·9 Determine the vertical radiation pattern of the array of problem 6·8 when it is spaced one wavelength above the ground. Assume a perfectly conducting ground surface.

6·10 Determine the vertical radiation pattern of a vertical stack of electrically small horizontal loop antennas spaced half a wavelength apart. The array consists of 4 antennas with the first antenna 5 wavelengths above the ground. Assume that the ground is a perfectly conducting surface and that the vertical radiation patterns of the individual loops are sine curves.

Chapter 7

WAVE GUIDES

7·1 Introduction

An ordinary receiving antenna receives only a very small fraction of the energy that the transmitting antenna is radiating. The reason for this inefficiency is that the ordinary transmitting antenna transmits the energy in a spherical wave which spreads out in all directions. The only major reflecting surface which may be encountered is the ground, but the wave spreads out in all other directions. Hence, for point-to-point reception, the energy has to be concentrated by means of an array of some type. However, even with the use of very directive arrays and reflectors for both the transmitting and receiving antennas, efficiency in the transmission of power is not yet obtainable through ordinary radiation and reception. Besides inefficiency, the transmission is not confined, and obstacles may be encountered, and bends may be necessary in the connecting path; all these make ordinary radiation and reception not usually desirable for power transmission.

However, material objects such as conducting surfaces will affect the path of radiation; a conducting surface would confine the electromagnetic waves to one side of the surface. A number of conducting surfaces may be used to confine an electromagnetic wave to a specified path. An analogous device is the speaking tube used with sound waves. Ordinarily sound waves spread out in all directions but they are reflected from solid objects. Even giving it directivity, by means of megaphones or similar devices, does not make for efficient power transmission. However, a speaking tube made of sound-reflecting walls will confine the sound to a path within the tube by having it bounce back and forth from wall to wall within the tube. Bends in the tube may be made provided they are not too sharp, and thus the sound may be carried for quite some distance with little attenuation. In a similar manner, a tube constructed of a conducting material, a good reflector for electromagnetic waves, may be used to confine the waves, from the source to the load, to a path within the tube. The source may be a transmitter and the load a radiator or the source may be an antenna and the load a receiver. These tubes, which may be square, rectangular, round, or any other useful cross section, are called wave guides.

Like transmission lines, wave guides are used for other purposes besides the transmission of power: They may be used as circuit elements and matching devices. They may also be used as measuring instruments or parts of measuring devices.

Variations of wave guides for use as antennas are called horns; they are very useful in the range where wave guides are applicable. A related device is the cavity resonator where the wave is confined to a specific volume adjusted to be resonant at the frequency employed. These cavity resonators are used as wave meters and circuit elements. More uses of confined wave devices will become evident as the theory pertaining to them is derived.

7·2 Method of Solution

The approach to the problem of the wave guide equations will be approximately the same as was used for the derivation of the radiation equations in Chapter 5. The method of solution consists of four parts:

1. Maxwell's equations are written down in the form most adaptable to the problem on hand.
2. The general conditions, applicable to the general problem, are substituted into the equations. These are conditions not for any specific problem but for the general case of the type of wave involved.
3. The equations obtained in part 2 are solved. It is not always possible with the knowledge available, and many have not yet been solved.
4. The boundary conditions for the *specific* case are then substituted into the equations obtained in part 3 and the resultant equations simplified and studied.

This type of approach* to a wave guide problem is straightforward; the final equations obtained will represent the actual wave equations found in the idealized case. The practical case sometimes differs from the ideal conditions, and care should be taken to check on these conditions before the results of part 4 are applied. Usually a study of the final equations will result in an understanding and, perhaps, an actual picture of what is taking place within the device being studied.

The field vectors within the wave guide are assumed to be a sinusoidal function of time; hence the alternating current form of Max-

* See R. I. Sarbacher and W. A. Edson, *Hyper and Ultrahigh Frequency Engineering*, New York, John Wiley and Sons, 1943, for an excellent analysis of wave guides and resonators using this approach.

well's equations is used. These equations, in the differential form as given in Equations 3·86 to 3·89, are repeated below:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{k_e \epsilon_0} \tag{7·1}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{7·2}$$

$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -j\omega k_m \mu_0 H_x \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -j\omega k_m \mu_0 H_y \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -j\omega k_m \mu_0 H_z \end{aligned} \tag{7·3}$$

$$\begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= (\sigma + j\omega k_e \epsilon_0) E_x \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= (\sigma + j\omega k_e \epsilon_0) E_y \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= (\sigma + j\omega k_e \epsilon_0) E_z \end{aligned} \tag{7·4}$$

The wave guide will be aligned in the coordinate system so that any wave traveling down the guide from the source to the load will travel in the x direction. Only the wave traveling in the plus x direction will be considered first; consequently, from the knowledge of the traveling waves which have been studied, the two vectors **E** and **H** will have an exponential multiplying factor containing t and x, namely, $e^{j\omega t - \Gamma x}$. Also, t and x will be assumed not to enter into the equations in any other manner. Γ is the propagation constant and ω is equal to 2π times the frequency, f, of the wave being propagated down the guide. Therefore,

$$\mathbf{E} = \mathbf{E}_0 e^{j\omega t - \Gamma x} \tag{7·5}$$

and

$$\mathbf{H} = \mathbf{H}_0 e^{j\omega t - \Gamma x} \tag{7·6}$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are vector functions.

The transverse electromagnetic wave (where both the electric intensity vector and the magnetic intensity vector are perpendicular to the direction of propagation—in this case the x direction) is not obtainable in the ordinary wave guide. The types obtainable are the

transverse electric wave (abbreviated TE or H), where the electric vector is perpendicular to the direction of propagation and the magnetic vector has a component in that direction, and the transverse magnetic wave (abbreviated TM or E), where the magnetic vector is perpendicular to the direction of propagation and the electric vector has a component in that direction. These two types will be treated separately as they are usually so used in practice.

7·3 THE TRANSVERSE ELECTRIC WAVE

In assuming a transverse electric wave in the wave guide shown in Figure 7·1, we assume that the x component of the electric intensity of the wave traveling in the plus x direction is zero. In other words, E_x is zero. This wave guide has the cross-sectional measurements of y_0 and z_0 and its length is in the x direction. From Equations 7·5 and 7·6, since the vectors are exponential functions of t and x, each of their components will be the same function of t and x. Hence the partial derivatives of the components of the vectors with respect to x, as noted in Equations 7·3 and 7·4, can be taken immediately as follows:

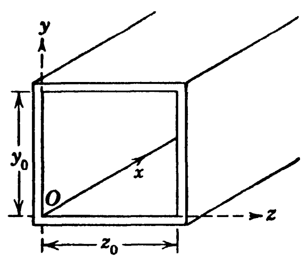


FIG. 7·1 A rectangular wave guide oriented in rectangular coordinates so that its length is in the x direction and its sides are parallel to the y and z axes.

$$\begin{aligned} \frac{\partial E_y}{dx} &= -\Gamma E_y \\ \frac{\partial E_z}{\partial x} &= -\Gamma E_z \\ \frac{\partial H_y}{dx} &= -\Gamma H_y \\ \frac{\partial H_z}{\partial x} &= -\Gamma H_z \end{aligned} \tag{7·7}$$

Using only an air dielectric, we know that k_e and k_m will both be equal to one. Substituting this, the condition that E_x is zero, and Equations 7·7, into Equations 7·3 and 7·4, we obtain

$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -j\omega\mu_0 H_x \\ \Gamma E_z &= -j\omega\mu_0 H_y \\ -\Gamma E_y &= -j\omega\mu_0 H_z \end{aligned} \tag{7·8}$$

$$\begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= 0 \\ \frac{\partial H_x}{\partial z} + \Gamma H_z &= (\sigma + j\omega\epsilon_0)E_y \\ -\Gamma H_y - \frac{\partial H_x}{\partial y} &= (\sigma + j\omega\epsilon_0)E_z \end{aligned} \tag{7·9}$$

Assuming also that the concentration of charge, ρ, within the dielectric of the guide is zero, we find that Equations 7·1 and 7·2 become

$$\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{7·10}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{7·11}$$

Equations 7·8 to 7·11 represent part 2 (page 210) in the solution of the problem. It represents the substitution of the general conditions, for the wave to be obtained, into Maxwell's equations. Part 3 calls for the solution of these equations, the solution being the expressions for the components of the field within the guide. Once these components are obtained, the final boundary conditions may be substituted into them as required by part 4.

The first step in the solution of the equations is to differentiate the second equation of Equations 7·9 with respect to z and the third equation of Equations 7·9 with respect to y:

$$\begin{aligned} \frac{\partial^2 H_x}{\partial z^2} + \Gamma \frac{\partial H_z}{\partial z} &= (\sigma + j\omega\epsilon_0) \frac{\partial E_y}{\partial z} \\ -\Gamma \frac{\partial H_y}{\partial y} - \frac{\partial^2 H_x}{\partial y^2} &= (\sigma + j\omega\epsilon_0) \frac{\partial E_z}{\partial y} \end{aligned} \tag{7·12}$$

Subtracting the second equation of Equations 7·12 from the first and gathering similar terms, we get

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} + \Gamma\left(\frac{\partial H_z}{\partial z} + \frac{\partial H_y}{\partial y}\right) = (\sigma + j\omega\epsilon_0)\left(-\frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial z}\right) \tag{7·13}$$

However, from the first equation of Equations 7·8 and the divergence Equation 7·11, we obtain

$$\begin{aligned} j\omega\mu_0 H_x &= -\frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial z} \\ -\frac{\partial H_x}{\partial x} &= \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = \Gamma H_x \end{aligned} \tag{7·14}$$

Substituting Equations 7·14 into Equation 7·13, we obtain

$$\frac{\partial^2 H_x}{\partial z^2} + \frac{\partial^2 H_x}{\partial y^2} + \Gamma^2 H_x = j\omega\mu_0(\sigma + j\omega\epsilon_0)H_x \tag{7·15}$$

Equation 7·15 is an equation in terms of only one component of the field, namely H_x.

Let us assume now that the leakage conductance of the air dielectric is zero. This means that σ in Equation 7·15 will be zero. Making this substitution into Equation 7·15 and simplifying, we obtain

$$\frac{\partial^2 H_x}{\partial z^2} + \frac{\partial^2 H_x}{\partial y^2} = -(\Gamma^2 + \omega^2\mu_0\epsilon_0)H_x \tag{7·16}$$

The next step is to solve Equation 7·16 in order to obtain the expression for H_x. It may be solved by the method of separation of variables. In this method, H_x is assumed to consist of the product of a function of z and a function of y. If $g_1(z)$ were the function of z and $g_2(y)$ were the function of y, the expression for H_x would be given by

$$H_x = [g_1(z)][g_2(y)] \tag{7·17}$$

For convenience, we may let Z represent $g_1(z)$ and Y represent $g_2(y)$ so that Equation 7·17 becomes

$$H_x = ZY \tag{7·18}$$

The Z and Y are merely symbolic representations of the two functions and are not meant to represent any impedance or admittance as they did in the case of transmission lines. The reason that Z and Y are used in this case, instead of any other symbols, is so that Z can be associated symbolically with a function of z and Y with a function of y. Substituting Equation 7·18 into Equation 7·16, we get

$$Z\frac{\partial^2 Y}{\partial y^2} + Y\frac{\partial^2 Z}{\partial z^2} = -(\Gamma^2 + \omega^2\mu_0\epsilon_0)YZ \tag{7·19}$$

and dividing through by ZY, we obtain

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -(\Gamma^2 + \omega^2\mu_0\epsilon_0) \tag{7·20}$$

This last equation shows that a function of Y (which in turn is a function of y) plus a function of Z (which in turn is a function of z) is equal to a constant. Inasmuch as y and z can vary independently, it is necessary that each of the functions on the left-hand side of Equation 7·20 be separately equal to a constant. The sum of the two separate constants will be equal to the constant on the right-hand side of Equa-

tion 7·20: Assume two constants A_1 and A_2, where

$$A_1 + A_2 = \Gamma^2 + \omega^2\mu_0\epsilon_0 \tag{7·21}$$

and

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -A_1 \tag{7·22}$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -A_2 \tag{7·23}$$

We notice the similarity between Equation 7·22 and Equation 7·23. Both these equations are well-known differential equations, the solutions of which are exponentials. Taking first Equation 7·22, we let

$$Y = Ce^{\eta y} \tag{7·24}$$

where C and η are constants. Substituting Equation 7·24 into Equation 7·22, we get

$$\frac{1}{Ce^{\eta y}}\eta^2 Ce^{\eta y} = -A_1 \tag{7·25}$$

which means that

$$\eta = \pm j\sqrt{A_1} \tag{7·26}$$

Since η has two values, Y will consist of the sum of two parts, each with its own constant. Calling the constants C_1 and C_2, we obtain

$$Y = C_1 e^{j\sqrt{A_1}y} + C_2 e^{-j\sqrt{A_1}y} \tag{7·27}$$

It is more convenient to express Equation 7·27 in terms of the sine and cosine of the angle $\sqrt{A_1}y$. This can be done because

$$\begin{aligned} C_1 e^{j\sqrt{A_1}y} &= C_1 \cos(\sqrt{A_1}y) + jC_1 \sin(\sqrt{A_1}y) \\ C_2 e^{-j\sqrt{A_1}y} &= C_2 \cos(\sqrt{A_1}y) - jC_2 \sin(\sqrt{A_1}y) \end{aligned} \tag{7·28}$$

Taking the sum of the two equations of Equations 7·28 and calling the new constants K_1 and K_2, we find that

$$Y = K_1 \sin \sqrt{A_1}y + K_2 \cos \sqrt{A_1}y \tag{7·29}$$

Since Equation 7·23 is similar to Equation 7·22, the solution for Z can be written down immediately:

$$Z = K_3 \sin(\sqrt{A_2}z) + K_4 \cos(\sqrt{A_2}z) \tag{7·30}$$

where K_3 and K_4 are constants.

The complete solution, as given by Equation 7·18, is the product of

Equations 7·29 and 7·30. The expression for H_x', therefore, is given by

$$H_x = YZ = [K_1 \sin(\sqrt{A_1}y) + K_2 \cos(\sqrt{A_1}y)] \\ [K_3 \sin(\sqrt{A_2}z) + K_4 \cos(\sqrt{A_2}z)] \quad (7\cdot31)$$

Expanding Equation 7·31, we obtain

$$H_x = K_1K_3 \sin(\sqrt{A_1}y)\sin(\sqrt{A_2}z) + K_1K_4 \\ \sin(\sqrt{A_1}y)\cos(\sqrt{A_2}z) + K_2K_3\cos(\sqrt{A_1}y) \\ \sin(\sqrt{A_2}z) + K_2K_4\cos(\sqrt{A_1}y)\cos(\sqrt{A_2}z) \quad (7\cdot32)$$

This is a long cumbersome solution but it should be used in the general case. However, it is possible to choose the boundary conditions in such a manner that all the terms on the right-hand side of Equation 7·32 except the term involving the product of two cosine terms will drop out. In other words, the constants K_1 and K_3 will both be zero. This is accomplished by choosing the correct orientation of the wave guide with respect to the coordinate axes.

In Figure 7·1, a rectangular wave guide is oriented so that one corner, a corner running lengthwise along the guide, coincides with the x axis. The guide is squared up with the axes so that the lengthwise sides are parallel to the y and z axes. We shall see that this orientation will make K_1 and K_3 both zero. The other constants may be complex numbers.

All components of **E** and **H** can now be expressed in terms of H_x. Substituting the second equation of Equations 7·8 into the third equation of Equations 7·9 and remembering that σ is zero, we obtain

$$\frac{\Gamma^2 E_z}{j\omega\mu_0} - \frac{\partial H_x}{\partial y} = (j\omega\epsilon_0)E_z \quad (7\cdot33)$$

Solving for E_z, we get

$$E_z = \frac{j\omega\mu_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{\partial H_x}{\partial y} \quad (7\cdot34)$$

Thus E is obtained in terms of H_x. The second equation of Equations 7·8, however, gives H_y in terms of E_z. Substituting for E_z from Equation 7·34 and solving for H_y, we obtain

$$H_y = -\frac{\Gamma}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{\partial H_x}{\partial y} \quad (7\cdot35)$$

which gives H_y in terms of H_x.

The expression for E_y is obtained by substituting for H_z in the second

equation of Equations 7·9 from the third equation of Equations 7·8:

$$\frac{\partial H_x}{\partial z} + \frac{\Gamma^2}{j\omega\mu_0} E_y = j\omega\epsilon_0 E_y \tag{7·36}$$

Solving for E_y, we obtain

$$E_y = -\frac{j\omega\mu_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{\partial H_x}{\partial z} \tag{7·37}$$

However, the third equation of Equations 7·8 expresses H_z in terms of E_y. By substituting Equation 7·37 into it and solving for H_z, H_z is obtained in terms of H_x:

$$H_z = -\frac{\Gamma}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{\partial H_x}{\partial z} \tag{7·38}$$

Since in this TE wave E_x is zero, Equation 7·38 completes the problem of expressing all the components in terms of H_x. Gathering all of the components together, we get

$$\begin{aligned} E_x &= 0 \\ E_y &= -\frac{j\omega\mu_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{\partial H_x}{\partial z} \\ E_z &= \frac{j\omega\mu_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{\partial H_x}{\partial y} \end{aligned} \tag{7·39}$$

and

$$\begin{aligned} H_x = {} & K_1K_3 \sin(\sqrt{A_1}y)\sin(\sqrt{A_2}z) + K_1K_4 \sin(\sqrt{A_1}y)\cos(\sqrt{A_2}z) \\ & + K_2K_3 \cos(\sqrt{A_1}y)\sin(\sqrt{A_2}z) + K_2K_4\cos(\sqrt{A_1}y)\cos(\sqrt{A_2}z) \end{aligned}$$

$$H_y = -\frac{\Gamma}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{\partial H_x}{\partial y}$$

$$H_z = -\frac{\Gamma}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{\partial H_x}{\partial z}$$

This is the completion of part 3 in the method of solution. We notice how all the components except H_x and E_x are equal to a constant times a derivative of H_x.

7·4 THE SUBSTITUTION OF BOUNDARY CONDITIONS FOR THE *TE* WAVES

The last part of the solution is to substitute the boundary conditions for a specific orientation of the guide into the general equations, Equations 7·39 and 7·40. Consider now the orientation shown in Figure

7·1. One side of the wave guide coincides with the y-equal-to-zero plane and another coincides with the z-equal-to-zero plane. For simplification, the sides are assumed to be perfect conductors. Since no component of the electric intensity vector, **E**, can exist on the surface of a perfect conductor, E_y is zero at z equal to zero and E_z is zero at y equal to zero.

It is the derivatives of H_x which determine the magnitudes of the components of the electric intensity vector; E_y is equal to a constant times the partial derivative of H_x with respect to z and E_z is equal to a constant times the partial derivative of H_x with respect to y. These two partial derivatives may be obtain by differentiating H_x, given in Equation 7·40. Performing this operation, we obtain

$$\begin{aligned}\frac{\partial H_x}{\partial z} &= K_1K_3\sqrt{A_2}\sin(\sqrt{A_1}y)\cos(\sqrt{A_2}z)\\ &- K_1K_4\sqrt{A_2}\sin(\sqrt{A_1}y)\sin(\sqrt{A_2}z)\\ &+ K_2K_3\sqrt{A_2}\cos(\sqrt{A_1}y)\cos(\sqrt{A_2}z)\\ &- K_2K_4\sqrt{A_2}\cos(\sqrt{A_1}y)\sin(\sqrt{A_2}z)\end{aligned} \tag{7·41}$$

and

$$\begin{aligned}\frac{\partial H_x}{\partial y} &= K_1K_3\sqrt{A_1}\cos(\sqrt{A_1}y)\sin(\sqrt{A_2}z)\\ &+ K_1K_4\sqrt{A_1}\cos(\sqrt{A_1}y)\cos(\sqrt{A_2}z)\\ &- K_2K_3\sqrt{A_1}\sin(\sqrt{A_1}y)\sin(\sqrt{A_2}z)\\ &- K_2K_4\sqrt{A_1}\sin(\sqrt{A_1}y)\cos(\sqrt{A_2}z)\end{aligned} \tag{7·42}$$

First let us consider the condition that E_y is zero at z equal to zero. This is true for any value of y. Let us assume now that neither A_1 nor A_2 is zero; it means that $\partial H_x/\partial z$ must be zero when the angle $(\sqrt{A_2}z)$ is zero while the angle $(\sqrt{A_1}y)$ assumes any value. In Equation 7·41 those terms, including the sine of $(\sqrt{A_2}z)$, will be zero when z is zero irrespective of the constant; but those terms containing the cosine of $(\sqrt{A_2}z)$ will be zero at z equal to zero for all values of y only if the constant in that term is zero. Consequently, for this specific orientation of the wave guide

$$K_1K_3 = K_2K_3 = 0 \tag{7·43}$$

Now let us consider the condition that E_z is zero at y equal to zero. This is true for any value of z. Hence $\partial H_x/\partial y$ is zero when the angle $(\sqrt{A_1}y)$ is zero while the angle $(\sqrt{A_2}z)$ assumes any value. In Equa-

tion 7·42, those terms containing the cosine of $(\sqrt{A_1}y)$ must have constants whose value is zero. Hence, for this orientation of the wave guide,

$$K_1K_3 = K_1K_4 = 0 \tag{7·44}$$

Only one remaining constant is left whose value is not zero, namely, K_2K_4. Calling this constant A in order to simplify writing the equations, we find that the expression for H_x reduces to a single term, the term containing the product of the two cosine terms:

$$H_x = A \cos(\sqrt{A_1}y) \cos(\sqrt{A_2}z) \tag{7·45}$$

Hence the two partial derivatives given in Equations 7·41 and 7·42 also reduce to a single term. Substituting Equation 7·45 into the equations for the components of the wave in the guide as given in Equations 7·39 and 7·40, we obtain

$$\begin{aligned} E_x &= 0 \\ E_y &= \frac{j\omega\mu_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0} A\sqrt{A_2} \cos(\sqrt{A_1}y) \sin(\sqrt{A_2}z) \\ E_z &= -\frac{j\omega\mu_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0} A\sqrt{A_1} \sin(\sqrt{A_1}y) \cos(\sqrt{A_2}z) \end{aligned} \tag{7·46}$$

and

$$\begin{aligned} H_x &= A \cos(\sqrt{A_1}y) \cos(\sqrt{A_2}z) \\ H_y &= \frac{\Gamma}{\Gamma^2 + \omega^2\mu_0\epsilon_0} A\sqrt{A_1} \sin(\sqrt{A_1}y) \cos(\sqrt{A_2}z) \\ H_z &= \frac{\Gamma}{\Gamma^2 + \omega^2\mu_0\epsilon_0} A\sqrt{A_2} \cos(\sqrt{A_1}y) \sin(\sqrt{A_2}z) \end{aligned} \tag{7·47}$$

The constants A_1 and A_2 can now be determined in a similar manner. The other two sides of the wave guide, namely, the side at z equal to z_0 and the side at y equal to y_0, cannot have any tangential electric field intensity vector exist on its surface. Consequently, E_y is zero when z is equal to z_0, which, from Equation 7·46, means that

$$\sin(\sqrt{A_2}z_0) = 0 \tag{7·48}$$

But the sine of an angle is equal to zero when the angle has integral multiples of π. In other words, the angle has to have values of 0, π, 2π, 3π, 4π, etc. It can also assume negative values theoretically—but not practically. Let us have m represent any integer. From

Equation 7·48, therefore,

$$\sqrt{A_2}z_0 = m\pi \tag{7·49}$$

Solving for the constant $\sqrt{A_2}$, we obtain

$$\sqrt{A_2} = \frac{m\pi}{z_0} \tag{7·50}$$

Similarly, from Equation 7·46, if E_z is zero when y is equal to y_0,

$$\sin(\sqrt{A_1}y_0) = 0 \tag{7·51}$$

Again this means that the angle can have only integral multiple values of π. Letting n also represent an integer value which can vary independently of the value of m, we obtain

$$\sqrt{A_1}y_0 = n\pi \tag{7·52}$$

and solving for the constant $\sqrt{A_1}$, we get

$$\sqrt{A_1} = \frac{n\pi}{y_0} \tag{7·53}$$

These two results can now be substituted into Equations 7·46 and 7·47 to obtain the final equations for the magnitudes of the field components within a wave guide pertaining to the transverse electric wave, usually referred to as the TE wave.

$$\begin{aligned} E_x &= 0 \\ E_y &= A\frac{j\omega\mu_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{m\pi}{z_0}\cos\left(\frac{n\pi}{y_0}y\right)\sin\left(\frac{m\pi}{z_0}z\right) \\ E_z &= -A\frac{j\omega\mu_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{n\pi}{y_0}\sin\left(\frac{n\pi}{y_0}y\right)\cos\left(\frac{m\pi}{z_0}z\right) \end{aligned} \tag{7·54}$$

and

$$\begin{aligned} H_x &= A\cos\left(\frac{n\pi}{y_0}y\right)\cos\left(\frac{m\pi}{z_0}z\right) \\ H_y &= A\frac{\Gamma}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{n\pi}{y_0}\sin\left(\frac{n\pi}{y_0}y\right)\cos\left(\frac{m\pi}{z_0}z\right) \\ H_z &= A\frac{\Gamma}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\frac{m\pi}{z_0}\cos\left(\frac{n\pi}{y_0}y\right) \end{aligned} \tag{7·55}$$

Thus the variations in amplitude of the components are sinusoidal functions of the coordinates, the functions being dependent on the integers m and n and on the dimensions of the guide.

The value of the propagation constant, Γ, still has to be determined. In Equations 7·5 and 7·6 all the components are multiplied by the exponential function of t and x, $e^{j\omega t-\Gamma x}$. This exponential determines the variation of the amplitude with time and position along the length of the guide. It is dependent on the value of Γ. Solving for Γ in Equation 7·21, we obtain

$$\Gamma = \pm\sqrt{A_1 + A_2 - \omega^2\mu_0\epsilon_0} \tag{7·56}$$

and substituting in the expressions for A_1 and A_2 as obtained in Equations 7·50 and 7·53, we find that

$$\Gamma = \pm\sqrt{\left(\frac{n\pi}{y_0}\right)^2 + \left(\frac{m\pi}{z_0}\right)^2 - \omega^2\mu_0\epsilon_0} \tag{7·57}$$

Thus Γ is also dependent on the value of the integers m and n and the cross-sectional size of the guide.

The constant A can be considered to be the general amplitude constant of the equations. It is dependent on the magnitude of the source voltage or other means used to excite the guide. Thus, if all other conditions remain constant and the source voltage is doubled, the only factor that changes in the equations is A, which also doubles. We see that it is a factor in all the amplitudes of the components so that their magnitudes are all directly dependent on the magnitude of A. A can be either real or complex depending on the configuration of the circuits, or elements, used in conjunction with the guide.

7·5 A DISCUSSION OF THE PROPAGATION CONSTANT Γ

Equations 7·5 and 7·6 are repeated, for convenience of reference, with the exponential function factored into two parts:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{j\omega t} e^{-\Gamma x} \\ \mathbf{H} &= \mathbf{H}_0 e^{j\omega t} e^{-\Gamma x} \end{aligned} \tag{7·58}$$

The exponent of e, dependent on the time, t, is an imaginary exponent, introducing a sinusoidal variation of the amplitude of the vectors with a variation in time, but not introducing any attenuation in the amplitude of those variations. It can be referred to as a phase factor. The exponent of e dependent on x, on the other hand, is determined by the value of the propagation function, Γ.

We see from Equation 7·57 that Γ can be either positive or negative depending on which root is chosen. Just as in the case of the transmission line solution or in the case of the radiation problem, it represents, when propagation takes place, two waves, one traveling in the plus x

direction and the other traveling in the minus x direction. Actually, each wave can be treated separately in a manner similar to the transmission line solution, where a reflection factor determines the magnitude and phase of the reflected wave. Hence the discussion will be limited to the wave traveling in the plus x direction since the wave traveling in the minus x direction will follow the same rules (the principles of reflection will be studied later). Therefore, only the positive root of the radical will be used in the discussion.

If Γ is real and positive, each of the vectors have a factor that would cause the magnitude to decrease as x is increased. In other words, the magnitude of the sinusoidal time variations is attenuated as x is increased but the phase of the variations do not vary with x. When Γ is imaginary it is only a phase factor and does not affect the peak magnitudes of the sinusoidal variations except to change their phase as x is varied.

Because of the minus sign, when the exponent Γ is a positive function, the increase of x will cause the phase angle of the components to lag more and more. This variation is the requisite of a wave traveling in the plus x direction. This wave resembles the wave propagated along a lossless transmission line (discussed in Chapter 1). In a lossy wave guide the propagation function will have both real and imaginary components introducing both phase shift and attenuation; this will be carefully discussed later.

Let us first assume that Γ is real. From Equation 7·57 we see that Γ is equal to the square root of a quantity involving the cross-sectional dimensions of the guide and the integers m and n. If the quantity under the radical sign is positive, Γ will be real. It will be positive when

$$\left(\frac{n\pi}{y_0}\right)^2 + \left(\frac{m\pi}{z_0}\right)^2 > \omega^2\mu_0\epsilon_0 \tag{7·59}$$

When this condition occurs, the wave will be attenuated along the guide and no true propagation will take place. It is interesting to note that if Equation 7·59 is true for certain values of m and n it will be true for all values of m and n, respectively, larger than those values.

Now let us assume that Γ is imaginary. This will take place when the quantity under the radical sign is negative. Factoring out a j term, we find that Γ becomes

$$\Gamma = j\sqrt{\omega^2\mu_0\epsilon_0 - \left[\left(\frac{n\pi}{y_0}\right)^2 + \left(\frac{m\pi}{z_0}\right)^2\right]} \tag{7·60}$$

where

$$\omega^2\mu_0\epsilon_0 > \left(\frac{n\pi}{y_0}\right)^2 + \left(\frac{m\pi}{z_0}\right)^2 \tag{7·61}$$

Thus Γ will, in this case, introduce only a phase shift in the components of the field within the guide and the wave will be propagated down the guide without attenuation. It should be noted that if Equation 7·61 is true for certain values of m and n, it will be true if m is kept constant and n is reduced, or if n is kept constant and m is reduced, or if they are both reduced together. Sometimes, though, it is possible to increase one of the integers and decrease the other and still retain the inequality of Equation 7·61.

Following this type of reasoning, we arrive at the critical case where

$$\omega^2\mu_0\epsilon_0 = \left(\frac{n\pi}{y_0}\right)^2 + \left(\frac{m\pi}{z_0}\right)^2 \tag{7·62}$$

This is the point at which attenuation stops and propagation takes place, or vice versa, depending on how it is approached. If all other factors are kept constant and f is increased, so that the value of ω in Equation 7·62 is increased, the condition of propagation shown in Equation 7·61 will be obtained. Similarly if the frequency is decreased the condition for attenuation shown in Equation 7·59 will be obtained. Thus Equation 7·62 determines a limiting frequency, for a specific guide and specific values of m and n, above which propagation takes place and below which attenuation takes place. Because of the existence of this cut-off frequency, the wave guide is sometimes referred to as a high pass filter. Variation of other factors in Equation 7·62 will also lead to similar results.

The factors m and n determine what is called the mode of transmission or the mode of the wave. Since there can be many types of TE waves, the type under discussion is usually noted as TE_{mn}. Putting in the m and n terms limits the discussion to that mode. There can be an infinite number of modes since there are an infinite number of combinations of integers. For instance, the mode of transmission where m is 3 and n is 2 would be noted as TE_{32}. Each mode is discussed separately. Only the first few are really important because the higher modes are seldom used.

7·6 THE TE_{01} WAVE

The mode wherein both m and n are zero does not exist; therefore, the lowest mode for the transverse electric wave is the one wherein one integer is zero and the other is one. Let us put m equal to zero and n equal to one. This type of wave is designated by TE_{01}. To obtain the critical (cut-off) frequency, f_{01}, these values are substituted into

Equation 7·62, where ω is taken equal to $2\pi f_{01}$.

$$(2\pi f_{01})^2 \mu_0 \epsilon_0 = \left(\frac{\pi}{y_0}\right) \tag{7·63}$$

Solving for f_{01}, we obtain

$$f_{01} = \frac{1}{2y_0\sqrt{\mu_0\epsilon_0}} \tag{7·64}$$

Thus the critical frequency is dependent only on the y dimension. However, one over the square root of the product of μ_0 and ϵ_0 is equal to v, the velocity of propagation of an electromagnetic wave through free space. Substituting this into Equation 7·64 and simplifying by replacing the velocity over the frequency by the wavelength of the wave in free space, designated by λ_{01}, we get

$$y_0 = \frac{\lambda_{01}}{2} \tag{7·65}$$

Hence the TE_{01} wave will be propagated if the y_0 dimension is greater than a free space half wavelength at the frequency being employed. Whether the wave will be propagated or not is independent of the other dimension.

By substituting m equal to 0 and n equal to 1 into Equation 7·60, 7·54, and 7·55, the equations representing the 01 type of wave are obtained. For completeness, the exponentials are also included.

$$\Gamma_{01} = j\sqrt{\omega^2\mu_0\epsilon_0 - \left(\frac{\pi}{y_0}\right)^2} \tag{7·66}$$

$$\begin{aligned} E_x &= 0 \\ E_y &= 0 \\ E_z &= -A\frac{j\omega\mu_0}{\Gamma_{01}^2 + \omega^2\mu_0\epsilon_0}\frac{\pi}{y_0}\sin\left(\frac{\pi}{y_0}y\right)e^{j\omega t - \Gamma_{01}x} \end{aligned} \tag{7·67}$$

$$\begin{aligned} H_x &= A\cos\left(\frac{\pi}{y_0}y\right)e^{j\omega t - \Gamma_{01}x} \\ H_y &= A\frac{\Gamma_{01}}{\Gamma_{01}^2 + \omega^2\mu_0\epsilon_0}\frac{\pi}{y_0}\sin\left(\frac{\pi}{y_0}y\right)e^{j\omega t - \Gamma_{01}x} \\ H_z &= 0 \end{aligned} \tag{7·68}$$

Equations 7·67 and 7·68 represent the magnitudes of the components of the electromagnetic field vectors within the wave guide for a TE_{01}

wave. We note that there is only one component of the vector **E** which is not zero and that the z component of **H** is zero.

By plotting the contour lines determined by Equations 7·67 and 7·68, the field patterns for the TE_{01} wave are obtained. They are shown in the two views of Figure 7·2. The electric lines are shown as solid lines, the magnetic lines are shown dotted. All the electric

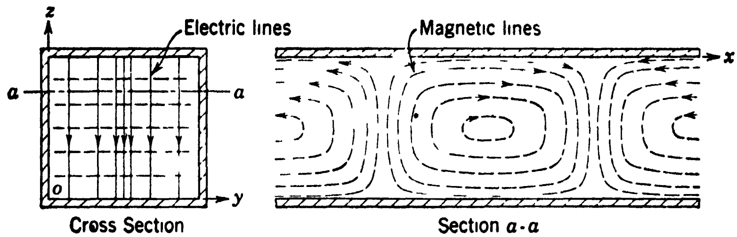


FIG. 7·2 Field patterns of the TE_{01} wave. The electric lines of force are shown solid while the magnetic lines of force are shown dotted.

lines are parallel to the z axis and terminate on the top and bottom of the wave guide as shown. The magnetic lines lie wholly within the z planes and are closed loops, each loop lying in a z plane. When the changing electric lines are pictured as a displacement current, we can see that the magnetic lines form closed loops around this current and lie in a plane at right angles to the current. With an increase in time, this whole configuration is shifted in the plus x direction, giving rise to the traveling wave conception which has been discussed. This type of wave is one of the most commonly used and will be discussed in more detail later on.

The TE_{10} wave is obtained by allowing m to equal 1 and n to equal 0. The results obtained are the same as for the TE_{01} wave except that the wave is shifted so that the electric lines are parallel to the y axis. The correct configuration would be obtained if the guide were rotated about its lengthwise axis so that the side that is on the z axis would be on the y axis of Figure 7·2 and the electric lines would be horizontal. The results obtained for the TE_{01} wave apply directly to the TE_{10} waves with the above modification.

EXAMPLE 7·1 Determine the critical frequencies of an air dielectric wave guide that has a rectangular cross section of inside dimensions 3 centimeters by 5 centimeters and is to be used with a TE_{01} wave.

There will be two critical frequencies dependent on the excitation orientation within the guide. If the electric vector in the wave being used is parallel to the 3-centimeters side, the critical frequency will be that frequency whose half wavelength in free space is 5 centimeters This will be 3,000 mega-

cycles. If the electric vector in the wave being used is parallel to the 5-centimeters side, the critical frequency will be that frequency whose half wavelength in free space is 3 centimeters. This frequency is equal to 5000 megacycles.

7·7 The TE_{11} wave

The TE_{11} mode is the next higher mode to be considered. It is the mode obtained when both m and n are put equal to 1. The first characteristic to be obtained is the critical frequency f_{11}. It is obtained by substituting $2\pi f_{11}$ for ω and 1 for m and for n in Equation 7·62. Solving for f_{11}, we get

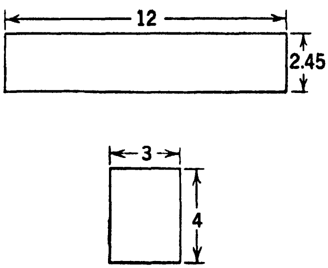


FIG. 7·3 Two different wave guides with the same cut-off frequency for the TE_{11} wave.

$$f_{11} = \frac{1}{2\sqrt{\mu_0\epsilon_0}}\sqrt{\frac{1}{y_0^2} + \frac{1}{z_0^2}} \tag{7·69}$$

which indicates that now the critical frequency is dependent on both the y_0 dimension and the z_0 dimension.

Since it is dependent on a sum involving both these dimensions, when one dimension of the guide is shortened the other may be elongated to obtain the same critical frequency. This is illustrated in Figure 7·3, where a guide 3 units by 4 units has the same cut-off, or critical, frequency as a guide 2.45 units by 12 units. It is true because

$$\frac{1}{(2.45)^2} + \frac{1}{(12)^2} = \frac{1}{(3)^2} + \frac{1}{(4)^2} \tag{7·70}$$

The equations for this type of wave are obtained by substituting 1 for m and for n in Equations 7·60, 7·54, and 7·55. Including the exponential factor, we obtain

$$\Gamma_{11} = j\sqrt{\omega^2\mu_0\epsilon_0 - \left[\left(\frac{\pi}{y_0}\right)^2 + \left(\frac{\pi}{z_0}\right)^2\right]} \tag{7·71}$$

$$E_x = 0$$

$$E_y = A\,\frac{j\omega\mu_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\,\frac{\pi}{z_0}\cos\left(\frac{\pi}{y_0}\,y\right)\sin\left(\frac{\pi}{z_0}\,z\right)e^{j\omega t-\Gamma_{11}x} \tag{7·72}$$

$$E_z = -A\,\frac{j\omega\mu_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0}\,\frac{\pi}{y_0}\sin\left(\frac{\pi}{y_0}\,y\right)\cos\left(\frac{\pi}{z_0}\,z\right)e^{j\omega t-\Gamma_{11}x}$$

$$H_x = A \cos\left(\frac{\pi}{y_0} y\right) \cos\left(\frac{\pi}{z_0} z\right)$$

$$H_y = A \frac{\Gamma}{\Gamma^2 + \omega^2 \mu_0 \epsilon_0} \frac{\pi}{y_0} \sin\left(\frac{\pi}{y_0} y\right) \cos\left(\frac{\pi}{z_0} z\right) e^{j\omega t - \Gamma_{11} x} \tag{7·73}$$

$$H_z = A \frac{\Gamma}{\Gamma^2 + \omega^2 \mu_0 \epsilon_0} \frac{\pi}{z_0} \cos\left(\frac{\pi}{y_0} y\right) \sin\left(\frac{\pi}{z_0} z\right) e^{j\omega t - \Gamma_{11} x}$$

In this case all the components exist except E_x. In Figure 7·4 are shown the field patterns obtained by plotting Equations 7·72 and 7·73 and thereby obtaining the magnitude and directions of the electro-

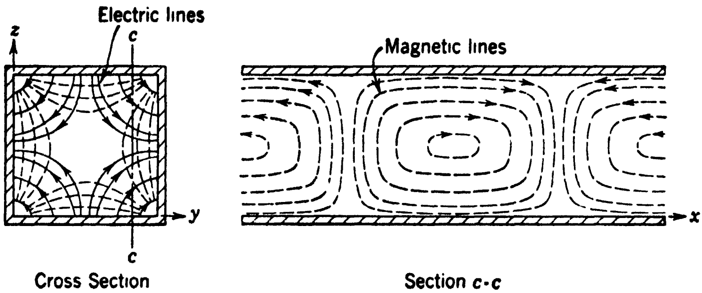


FIG. 7·4 The field patterns for the TE_{11} wave at one instant of time.

magnetic vectors within the guide. Notice that now the electric lines terminate on all four sides of the guide in a symmetrical manner. The magnetic loops are still perpendicular to the electric lines but because of the curvature of those lines the loops no longer lie in a plane surface.

EXAMPLE 7·2 Determine the critical frequency of an air dielectric wave guide which has a rectangular cross section with inside dimensions of 3 centimeters by 4 centimeters and is to be used with a TE_{11} wave.

These dimensions have to be substituted into Equation 7·69, the equation for the critical frequency. A factor of 10^{-2} has to be used with each of the dimensions to convert them to meters:

$$f_{11} = \frac{3 \times 10^8}{2} \sqrt{\frac{1}{(3 \times 10^{-2})^2} + \frac{1}{(4 \times 10^{-2})^2}}$$

$$f_{11} = 6250 \text{ megacycles} \qquad \textit{Ans.}$$

7·8 HIGHER MODES OF THE *TE* WAVE

The same method of analysis used in the preceding two sections may be applied to the higher modes of the *TE* wave. The values of m and

n for these higher modes are substituted into Equations 7·60, 7·54, and 7·55, resulting in the equations applicable to these modes for a rectangular wave guide.

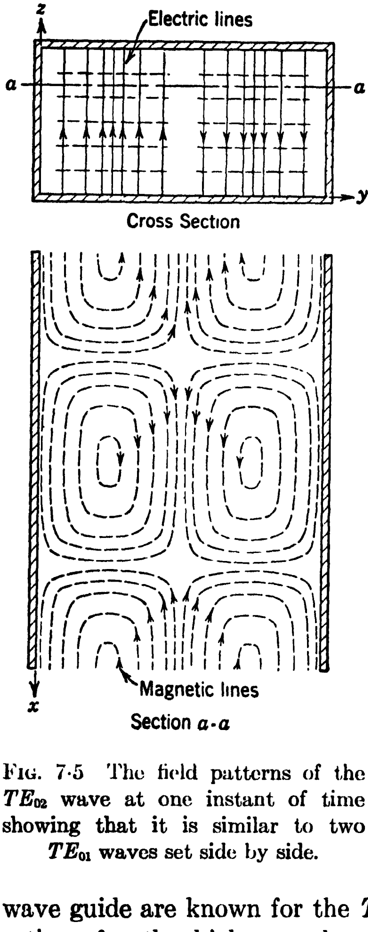


FIG. 7·5 The field patterns of the TE_{02} wave at one instant of time showing that it is similar to two TE_{01} waves set side by side.

It was pointed out previously that if a certain wave guide will transmit a definite mode it will pass all lower ones also. The method of excitation will determine the mode or groups of modes that will be transmitted.

For a picture of the configuration of the electric and magnetic lines of force within the guide, it is possible to use combinations of several TE_{01} waves or several TE_{11} waves. For instance, a TE_{02} wave would consist of two TE_{01} waves set side by side as shown in Figure 7·5. Similarly, higher modes of the TE_{0n} type would consist of combinations of the TE_{01} wave set side by side, n repetitions being used for a TE_{0n} wave.

When m is not zero, such as in the TE_{12} wave, the resultant configuration would be similar to two TE_{11} waves set side by side. Similarly, the TE_{32} wave would be similar to the configuration obtained by placing six TE_{11} waves together in a rectangular fashion, two by three. Thus once the configurations of the electric and magnetic lines within a rectangular wave guide are known for the TE_{01} and the TE_{11} waves, the configurations for the higher modes are deducible from them.

7·9 THE TRANSVERSE MAGNETIC WAVE

The other common type of wave which is propagated down the rectangular wave guide is the transverse magnetic wave, denoted symbolically by either TM or E, with the proper subscript to specify the mode being employed. The conditions for the derivation of the equations for the transverse magnetic wave are the same as those for the transverse electric wave except that the longitudinal component

of magnetic intensity, H_x, instead of the electric intensity component, E_x, is assumed to be zero.

Assuming again that the wave will travel along the x axis, that the propagation function is Γ, and that the wave will have an angular phasor velocity of ω, we find that Equations 7·6 and 7·7 also apply. When the wave conditions are substituted into Equations 7·3 and 7·4, the equations become

$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= 0 \\ \frac{\partial E_x}{\partial z} + \Gamma E_z &= -j\omega\mu_0 H_y \\ -\Gamma E_y - \frac{\partial E_x}{\partial y} &= -j\omega\mu_0 H_z \end{aligned} \tag{7·74}$$

and

$$\begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= (\sigma + j\omega\epsilon_0)E_x \\ \Gamma H_z &= (\sigma + j\omega\epsilon_0)E_y \\ -\Gamma H_y &= (\sigma + j\omega\epsilon_0)E_z \end{aligned} \tag{7·75}$$

Perfect conductors for the sides of the wave guides and lossless dielectric material for the interior of the guide are again assumed. Consequently, σ in the above equations will be zero, and the final equations for the wave may be obtained in a manner similar to the derivation employed for the *TE* waves.

In this case of the *TM* wave, two waves are also obtained: one traveling in the plus x direction and the other in the minus x direction. Only the plus wave is shown in the equations, and the minus wave is similar except that the sign of the propagation factor is changed. The final equations for this type of wave are:

$$\Gamma = j\sqrt{\omega^2\mu_0\epsilon_0 - \left[\left(\frac{n\pi}{y_0}\right)^2 + \left(\frac{m\pi}{z_0}\right)^2\right]} \tag{7·76}$$

$$\begin{aligned} E_x &= A \sin\left(\frac{n\pi}{y_0} y\right) \sin\left(\frac{m\pi}{z_0} z\right) \epsilon^{j\omega t - \Gamma x} \\ E_y &= -A \frac{\Gamma}{\Gamma^2 + \omega^2\mu_0\epsilon_0} \frac{n\pi}{y_0} \cos\left(\frac{n\pi}{y_0} y\right) \sin\left(\frac{m\pi}{z_0} z\right) e^{j\omega t - \Gamma x} \\ E_z &= -A \frac{\Gamma}{\Gamma^2 + \omega^2\mu_0\epsilon_0} \frac{m\pi}{z_0} \sin\left(\frac{n\pi}{y_0} y\right) \cos\left(\frac{m\pi}{z_0} z\right) e^{j\omega t - \Gamma x} \end{aligned} \tag{7·77}$$

$$H_x = 0$$

$$H_y = A \frac{j\omega\epsilon_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0} \frac{m\pi}{z_0} \sin\left(\frac{n\pi}{y_0} y\right) \cos\left(\frac{m\pi}{z_0} z\right) e^{j\omega t - \Gamma x} \tag{7·78}$$

$$H_z = -A \frac{j\omega\epsilon_0}{\Gamma^2 + \omega^2\mu_0\epsilon_0} \frac{n\pi}{y_0} \cos\left(\frac{n\pi}{y_0} y\right) \sin\left(\frac{m\pi}{z_0} z\right) e^{j\omega t - \Gamma x}$$

In these equations m and n are again integers that determine the mode of the wave. We see that the expression for Γ, given in Equation 7·76, is the same as the expression for the propagation constant for the *TE* modes. The discussion in section 7·5, therefore, applies also to this type of wave. There is one other major difference between the two types of waves besides the distinction that one has an x component of **E** and the other an x component of **H**. In the *TM* mode, E_x is determined by the product of two sine terms. The sine of zero is zero so that E_x will be identically zero if either m or n is zero. Consequently, neither m nor n can be zero if the wave is to exist; the 00, 01, and 10 modes do not exist for the *TM* type and the lowest existing *TM* mode is the TM_{11} wave.

7·10 The TM_{11} waves

The equations for the TM_{11} mode are obtained by substituting one for both m and n in Equations 7·76, 7·77, and 7·78. The cut-off or critical frequency for this type of wave is the same as for the transverse electric waves and is obtained by substituting into Equation 7·62 and solving for the critical frequency. For the TM_{11} mode the result is the same as for the TE_{11} mode, as given in Equation 7·69. Calling the critical frequency f_{11}, we get

$$f_{11} = \frac{1}{2\sqrt{\mu_0\epsilon_0}} \sqrt{\frac{1}{{y_0}^2} + \frac{1}{{z_0}^2}} \tag{7·79}$$

The behavior of these modes is the same as for the *TE* modes. Below the critical frequency, attenuation and no propagation take place whereas, above the critical frequency, propagation takes place.

The equations for the field components for the TM_{11} wave are

$$\Gamma = j\sqrt{\omega^2\mu_0\epsilon_0 - \left[\left(\frac{\pi}{y_0}\right)^2 + \left(\frac{\pi}{z_0}\right)^2\right]} \tag{7·80}$$

$$E_x = A \sin\left(\frac{\pi}{y_0} y\right) \sin\left(\frac{\pi}{z_0} z\right) e^{j\omega t - \Gamma x}$$

$$E_y = -A \frac{\Gamma}{\Gamma^2 + \omega^2 \mu_0 \epsilon_0} \frac{\pi}{y_0} \cos\left(\frac{\pi}{y_0} y\right) \sin\left(\frac{\pi}{z_0} z\right) e^{j\omega t - \Gamma x} \tag{7·81}$$

$$E_z = -A \frac{\Gamma}{\Gamma^2 + \omega^2 \mu_0 \epsilon_0} \frac{\pi}{z_0} \sin\left(\frac{\pi}{y_0} y\right) \cos\left(\frac{\pi}{z_0} z\right) e^{j\omega t - \Gamma x}$$

$$H_x = 0$$

$$H_y = A \frac{j\omega\epsilon_0}{\Gamma^2 + \omega^2 \mu_0 \epsilon_0} \frac{\pi}{z_0} \sin\left(\frac{\pi}{y_0} y\right) \cos\left(\frac{\pi}{z_0} z\right) e^{j\omega t - \Gamma x} \tag{7·82}$$

$$H_z = -A \frac{j\omega\epsilon_0}{\Gamma^2 + \omega^2 \mu_0 \epsilon_0} \frac{\pi}{y_0} \cos\left(\frac{\pi}{y_0} y\right) \sin\left(\frac{\pi}{z_0} z\right) e^{j\omega t - \Gamma x}$$

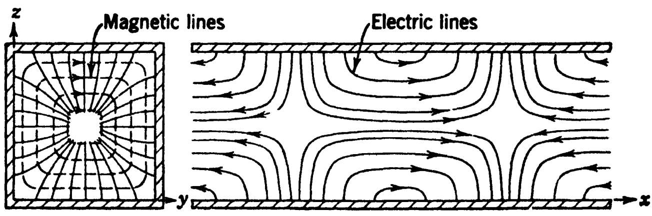


FIG. 7·6 The field patterns of the TM_{11} wave at one instant of time. The magnetic lines are now contained in the cross-sectional plane while the electric lines extend down the length of the guide.

In Figure 7·6 are shown the field configurations for a TM_{11} wave where the electric lines are shown as solid and the magnetic lines as dotted. The magnetic lines are closed loops in the plane perpendicular to the length of the guide. This agrees with the name of the wave, a *transverse magnetic wave.* The electric lines terminate on the sides of the guide and stretch out toward its middle and down its length. It is the displacement current of these changing electric field intensities that the magnetic loops enclose.

7·11 HIGHER MODES OF THE *TM* WAVE

The higher mode equations for the *TM* wave are obtained in the same manner as for the TM_{11} wave except that larger integers are substituted for either m or n, or both, into Equations 7·76, 7·77, and 7·78. Similarly, Equation 7·62 can be used to obtain the critical frequencies.

In regard to the configurations of the fields for these higher modes, Figure 7·7 shows the configuration obtained by plotting the equations for a TM_{21} wave. The equations were obtained by substituting two for m and one for n in the equations for the TM modes. Comparing these patterns with the patterns obtained for the TM_{11} mode, it can

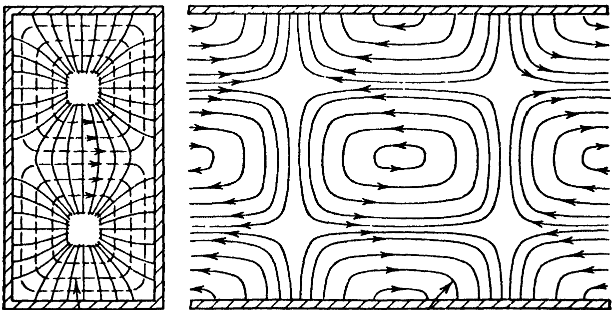


FIG. 7·7 The field patterns of the TM_{12} mode shown at one instant of time; the patterns resemble two TM_{11} waves placed one above the other.

be seen how this wave resembles two TM_{11} waves placed one above the other. This leads into a simple way of picturing the configurations for the higher TM modes. A TM_{22} wave would resemble four TM_{11} waves set in a square two by two. A TM_{32} wave would resemble six TM_{11} waves set in a rectangle three by two. Similarly the configurations for the higher modes can be pictured as a rectangle made up of TM_{11} waves set side by side.

7·12 Group and phase velocity*

In lossless transmission lines and in radiation through a dielectric, or any other lossless medium, the velocity of propagation of the wave is dependent on the comparative dieletric constant, k_e, and the comparative permeability constant, k_m. The velocity, v, is given by

$$v = \frac{1}{\sqrt{k_e k_m \mu_0 \epsilon_0}} \tag{7·83}$$

where ϵ_0 and μ_0 are the dielectric constant and the permeability constant, respectively, of free space. It was at this velocity v that both

* See H. H. Skilling, *Fundamentals of Electric Waves*, New York, John Wiley & Sons, 1942, for an excellent discussion of this subject.

the wave itself and the envelope of the wave, the modulation put on it, advanced. In other words, if the instantaneous wave form were plotted at successive instants of time it would be found that under normal conditions the peaks of the high frequency wave, usually called the carrier, and the peaks of the envelope containing the intelligence both advance at the speed v.

These two velocities will differ if, and only if, the carrier velocity is a function of frequency. In hollow wave guides, even though they are assumed lossless, the two velocities are not the same inasmuch as the carrier velocity is a function of frequency. The speed at which the envelope of the wave progresses is called the group velocity, v_g. Inside a guide this group velocity is slower than the speed of light. It is dependent on the size of the guide and the frequency being used. The speed at which a peak of an individual high frequency wave progresses is called the phase velocity, v_p, and is greater than the speed of light.

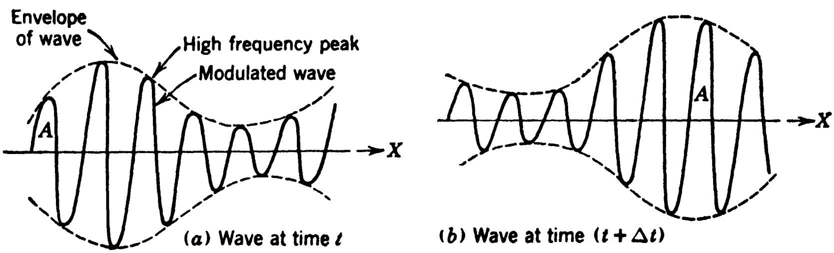


FIG. 7·8 Two figures illustrating the relationship of phase velocity to group velocity. Wave A in the high frequency group progresses forward under the envelope as the wave is propagated down the guide.

These velocities may be a little better appreciated by the two waves shown in Figure 7·8. In (a) is shown a modulated wave drawn by plotting the values of the electric field intensity, along the center of the guide, at one instant of time, t. One of the high frequency waves has been marked with the letter A. The envelope of the wave, which contains the intelligence, is shown dotted. A little later in time, $t + \Delta t$, the wave is shown plotted again in (b). The individual waves will move at a velocity greater than that of light and the envelope will move at a velocity slower than that of light. Hence the individual waves will move forward in their position in the envelope so that the marked wave, A, will occur at a different point in the envelope, the new position being further advanced down the envelope, occurring after the peak of the modulation envelope rather than before the peak as in (a).

The phase velocity is obtained by letting

$$\Gamma_{mn} = j\beta_{mn} \tag{7·84}$$

so that

$$\beta_{mn} = \sqrt{\omega^2\mu_0\epsilon_0 - \left[\left(\frac{n\pi}{y_0}\right)^2 + \left(\frac{m\pi}{z_0}\right)^2\right]} \tag{7·85}$$

The phase velocity, v_p, is given by

$$v_p = \frac{\omega}{\beta_{mn}}. \tag{7·86}$$

and substituting for β_{mn} from Equation 7·85, we get

$$v_p = \frac{1}{\sqrt{\mu_0\epsilon_0 - \frac{1}{\omega^2}\left[\left(\frac{n\pi}{y_0}\right)^2 + \left(\frac{m\pi}{z_0}\right)^2\right]}} \tag{7·87}$$

Comparing Equation 7·87 with the velocity in free space given by Equation 7·83 when both k_e and k_m equal one, we can see that the denominator in Equation 7·87 is always smaller than the denominator in Equation 7·83 for all values of m and n other than 00. This means that v_p will always be greater than the velocity of light and will depend on the cross-sectional dimensions of the guide and the mode being used.

The group velocity, v_g, is given by

$$v_g = \frac{{v_0}^2}{v_p} \tag{7·88}$$

where v_0 is the velocity of light in free space. Thus the product of the group velocity and the phase velocity, in any wave guide and for a specific mode, is equal to the square of the velocity of light in free space.

A picture of what is taking place within the guide may be obtained by examining the TE_{01} wave. The phase velocity for this mode $(v_p)_{01}$ is obtained by substituting m equal to zero and n equal to one in Equation 7·87:

$$(v_p)_{01} = \frac{1}{\sqrt{\mu_0\epsilon_0 - \left(\frac{\pi}{\omega y_0}\right)^2}} \tag{7·89}$$

This velocity, of course, is greater than the velocity of light. It may be explained by referring to Figure 7·9. In the figure is shown a wave traveling in the direction ab indicated by the wave fronts which are

perpendicular to this line. Imagine, however, that by some accident the velocity were measured along the line *ac*. This means that in the time that a wave front has traveled from *a* to *b* it will have seemed to travel a greater distance from *a* to *c*. Let us call the angle between *ab* and *ac*, θ; then the velocity measured along *ac* will be equal to one over the cosine of θ times the true velocity, one over cosine θ determining the difference in the lengths involved.

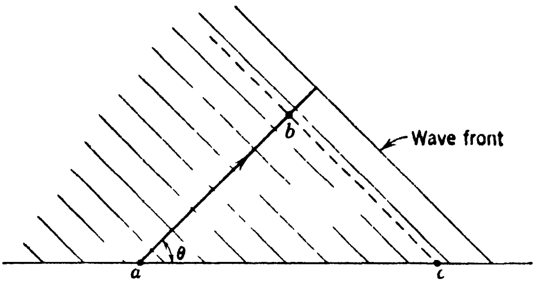


FIG. 7·9 Velocity measured at an angle to the direction of travel of the wave.

An explanation of the velocities within the guide would be that this method of measurement is being used inside the guide; indicating, that the wave within the guide is actually traveling at an angle to the x coordinate, being reflected from side to side by the walls of the guide. This would mean that when the velocity is measured along the length of the guide it is actually being measured at an angle to the direction of its ray, the direction of propagation of a "plane" wave within the guide.

Following this reasoning, we can obtain the angle to the x axis which the TE_{01} wave supposedly makes when it is traveling down the guide. To obtain the angle, the expression for E_z as given in the Equations 7·67 (the equations for the TE_{01} wave) is employed. Grouping all the constants together, factoring out the j, and calling the rest B, we find that the equation reduces to

$$E_z = -jB \sin\left(\frac{\pi}{y_0} y\right) e^{j\omega t - j\left[\sqrt{\omega^2\mu_0\epsilon_0 - \left(\frac{\pi}{y_0}\right)^2}\right]x} \tag{7·90}$$

where the expression for Γ_{01} has been substituted for Γ_{01} from Equation 7·66. However, the sine term may be replaced by the difference between two exponential terms, where

$$\sin\left(\frac{\pi}{y_0} y\right) = \frac{e^{j\left(\frac{\pi}{y_0} y\right)} - e^{-j\left(\frac{\pi}{y_0} y\right)}}{2j} \tag{7·91}$$

Substituting this into Equation 7·90, we get

$$E_z = -\frac{B}{2}\left\{e^{j\left(\frac{\pi}{y_0}y\right)} - e^{-j\left(\frac{\pi}{y_0}y\right)}\right\}\left\{e^{j\omega t - j\omega\left[\sqrt{\omega^2\mu_0\epsilon_0 - \left(\frac{\pi}{\omega y_0}\right)^2}\right]x}\right\} \tag{7·92}$$

Simplifying, we obtain

$$\begin{aligned} E_z = -\frac{B}{2}\Bigg\{ & e^{j\omega\left[t-\left(\sqrt{\mu_0\epsilon_0-\left(\frac{\pi}{\omega y_0}\right)^2}\right)x+\left(\frac{\pi}{\omega y_0}\right)y\right]} \\ & - e^{j\omega\left[t-\left(\sqrt{\mu_0\epsilon_0-\left(\frac{\pi}{\omega y_0}\right)^2}\right)x-\left(\frac{\pi}{\omega y_0}\right)y\right]}\Bigg\} \end{aligned} \tag{7·93}$$

Each of the exponential functions is considered to be a separate wave with a multiplying constant of $B/2$. The negative sign in front of one

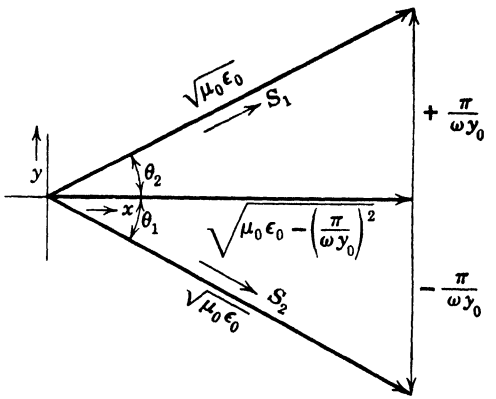


FIG. 7·10 The resolution of the TE_{01} wave into two waves traveling at angles θ_1 and θ_2 to the lengthwise axis of the guide.

of the waves means that one wave will be 180° out of phase with the other wave. This is correct for a reflected wave wherein the electric vector is parallel to the reflecting surface. The phase of each of these waves will now vary with both x and y so that the waves are traveling at an angle to the x axis in the xy plane. We see that the x components of the resultant β's are the same but that y components differ in sign.

In Figure 7·10 are shown the resultant directions for the two waves. Calling one direction S_1 and the other S_2, as shown, we see that the resultant phase constant of both waves comes out to be the same. It is obtained by taking the square root of the sum of the squares of the two components of each phase constant and results in the phase constant for free space, $\sqrt{\mu_0\epsilon_0}$. Thus we can consider two waves to be in the guide for the positive x direction of propagation: one travel-

ing in the direction S_1 at an angle θ_1 to the x axis and the other traveling in the direction S_2 at an angle θ_2 to the x axis. Both are traveling with the velocity of light. The angles may be obtained directly from Figure 7·10, where

$$-\theta_1 = \theta_2 = \tan^{-1} \frac{1}{\sqrt{\mu_0\epsilon_0\left(\dfrac{\omega y_0}{\pi}\right)^2 - 1}} \tag{7·94}$$

It is interesting to note that, for any frequency, as the dimension y_0 decreases the angle becomes larger and larger until at the critical frequency it becomes 90°.

Because the wave is traveling back and forth, the energy has to travel a greater distance than if the wave were traveling in a straight line. This increase in distance can be considered the reason why the group velocity is less than the speed of light. It represents the added time necessary because of the greater distance the energy in the wave has to travel.

EXAMPLE 7·3 Determine the phase and group velocities of a rectangular wave guide having inside cross-sectional measurements of 3 centimeters by 4 centimeters and used with a TM_{11} wave at a frequency of 10,000 megacycles.

For the phase velocity, the dimensions and the frequency are substituted Equation 7·87, with m and n equal to one:

$$v_p = \frac{1}{\sqrt{0.111 \times 10^{-16} - \dfrac{1}{4\pi^2 \times 10^{20}}\left[\left(\dfrac{\pi}{0.03}\right)^2 + \left(\dfrac{\pi}{0.05}\right)\right]}}$$

$$v_p = \frac{1}{\sqrt{0.111 \times 10^{-16} - 0.038 \times 10^{-16}}}$$

$$v_p = 3.8 \times 10^8 \text{ meters per second} \qquad \textit{Ans. (a)}$$

The group velocity is given by Equation 7·88:

$$v_g = \frac{9 \times 10^{16}}{3.8 \times 10^8}$$

$$v_g = 2.36 \times 10^8 \text{ meters per second} \qquad \textit{Ans. (b)}$$

7·13 CYLINDRICAL WAVE GUIDES

Unlike the wave guides discussed in the previous sections, a wave guide does not have to be rectangular in cross section but can be of very many other cross-sectional shapes. Among the commonest of the other shapes is the circular cross section. This type is known as a cylindrical wave guide.

It is much more convenient for the problems involving cylindrical shapes to use cylindrical coordinates. In Figure 7·11 is shown the relationship between cylindrical coordinates and Cartesian coordinates. The z axis is common to both systems for this particular orientation but r and ϕ are used instead of x and y. The distance r is the radial distance of the point under consideration from the z axis and ϕ is the angle which a projection of the perpendicular line (joining the point under consideration to the z axis) on the xy plane makes with the positive x axis.

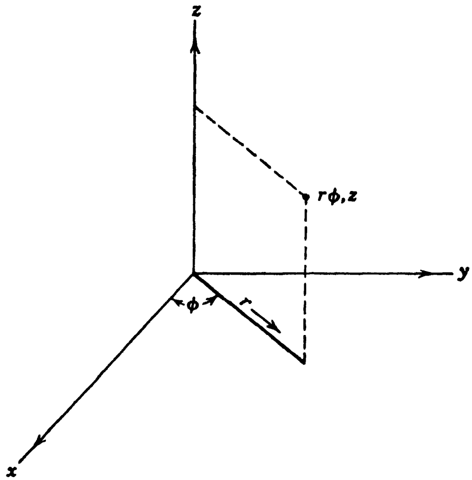


FIG. 7·11 Cylindrical coordinates using r, ϕ, and z, showing their relationship to Cartesian coordinates.

Maxwell's equations can also be expressed in cylindrical coordinates. Two methods of obtaining them are possible. One is to take the equations in Cartesian coordinates and to convert them to cylindrical coordinates. The other is to start again from the known integral relationships and to develop anew Maxwell's equations in cylindrical coordinates. This latter method would follow along the lines given in Chapter 3 except that cylindrical coordinates would be used. Either of these methods will result in the following equations for the alternating current form of Maxwell's equations in cylindrical coordinates:

$$\frac{1}{r}\left(\frac{\partial(rD_r)}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial D_\phi}{\partial \phi}\right)+\frac{\partial D_z}{\partial z}=\rho \qquad (7\cdot95)$$

$$\frac{1}{r}\left(\frac{\partial(rB_r)}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial B_\phi}{\partial \phi}\right)+\frac{\partial B_z}{\partial z}=0 \qquad (7\cdot96)$$

$$\frac{1}{r}\left(\frac{\partial H_z}{\partial \phi}\right) - \frac{\partial H_\phi}{\partial z} = (\sigma + j\omega k_e\epsilon_0)E_r$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = (\sigma + j\omega k_e\epsilon_0)E_\phi \qquad (7{\cdot}97)$$

$$\frac{1}{r}\left(\frac{\partial (rH_\phi)}{\partial r} - \frac{\partial H_r}{\partial \phi}\right) = (\sigma + j\omega k_e\epsilon_0)E_z$$

$$\frac{1}{r}\left(\frac{\partial E_z}{\partial \phi}\right) - \frac{\partial E_\phi}{\partial z} = -j\omega k_m\mu_0 H_r$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -j\omega k_m\mu_0 H_\phi \qquad (7{\cdot}98)$$

$$\frac{1}{r}\left(\frac{\partial (rE_\phi)}{\partial r} - \frac{\partial E_r}{\partial \phi}\right) = -j\omega k_m\mu_0 H_z$$

where E_r, E_ϕ, and E_z are the components of **E**; and H_r, H_ϕ, and H_z are the components of **H**. The other symbols in the equations have been previously defined.

In Figure 7·12 is shown the orientation of the coordinate axis with relation to the cylindrical guide to be used for the analysis. The z axis is oriented so that it becomes the centerline of the guide and ϕ the angle between a horizontal plane and the plane determined by the point under consideration and the z axis.

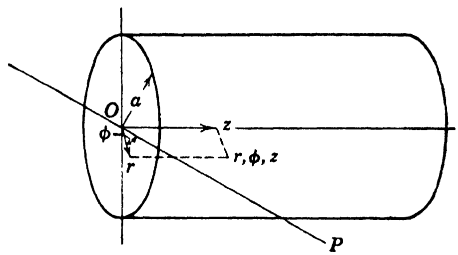


Fig. 7·12 Cylindrical coordinates for use with cylindrical wave guides; the z axis is oriented along the center line of the guide.

Let us assume that the guide dielectric is free space so that both k_e and k_m will be one and σ will be zero. Also let us assume that when propagation takes place it will take place only in the z direction. Hence

$$\begin{aligned} \mathbf{E} &= \mathbf{E}' e^{j\omega t - \Gamma z} \\ \mathbf{H} &= \mathbf{H}' e^{j\omega t - \Gamma z} \end{aligned} \qquad (7{\cdot}99)$$

which means that the derivative of any component with respect to z will be equal to $-\Gamma$ times that component. This derivation follows the same method as was used for derivation of the rectangular wave guide equations. The two curl equations, Equations 7·97 and 7·98

may now be simplified for this application:

$$\frac{1}{r}\left(\frac{\partial H_z}{\partial \phi}\right) + \Gamma H_\phi = j\omega\epsilon_0 E_r$$

$$-\Gamma H_r - \frac{\partial H_z}{\partial r} = j\omega\epsilon_0 E_\phi \tag{7·100}$$

$$\frac{1}{r}\left(\frac{\partial (rH_\phi)}{\partial r} - \frac{\partial H_r}{\partial \phi}\right) = j\omega\epsilon_0 E_z$$

and

$$\frac{1}{r}\left(\frac{\partial E_z}{\partial \phi}\right) + \Gamma E_\phi = -j\omega\mu_0 H_r$$

$$-\Gamma E_r - \frac{\partial E_z}{\partial r} = -j\omega\mu_0 H_\phi \tag{7·101}$$

$$\frac{1}{r}\left(\frac{\partial (rE_\phi)}{\partial r} - \frac{\partial H_r}{\partial \phi}\right) = -j\omega\mu_0 H_z$$

Two fundamental types of waves are found to exist as in the case of the rectangular guides: one in which there is a longitudinal component (in this case a z component) of the electric intensity vector and no longitudinal component of the magnetic intensity vector; and the other in which there is a longitudinal component of the magnetic intensity vector and no longitudinal component of the electric intensity vector. The one containing the longitudinal component of **H** is known as the transverse electric wave, abreviated *TE* or *H* wave, and the wave containing the longitudinal component of **E** is known as the transverse magnetic wave, abreviated *TM* or *E* wave. This notation conforms with the designations used in rectangular guides. Again there can be many modes; consequently, subscripts are used to designate the particular mode being discussed.

The same procedure as used for the rectangular guides is followed in the solution of these equations. For the *TE* waves E_z in Equations 7·100 and 7·101 is equated to zero and an equation involving only H_z is obtained by multiple substitutions. Separation of variables is employed and the final equations necessary for the solution for H_z are obtained. Unfortunately the solution is quite complex and involves Bessel functions. However, once the solution for H_z is obtained, the other components are expressed in terms of H_z which, in combination with the solution for Γ, yield a complete set of equations. The final step is taken by assuming that the guide wall is a perfect conductor so that the tangent **E** at the surface is zero. Substituting in this

condition, we obtain the final equations. For the *TM* wave the same procedure is followed with H_z assumed to be zero, and the solution for E_z is obtained.

7·14 Bessel functions*

Bessel functions are solutions of what is known as Bessel's equation. Bessel's equation is given by

$$\rho^2 \frac{\partial^2 Q}{\partial \rho^2} + \rho \frac{\partial Q}{\partial \rho} + (\rho^2 - n^2) Q = 0 \tag{7·102}$$

Q is the function that is desired, n is the order of the equation, and ρ is the argument. There are two linearly independent solutions inasmuch as it is a second-order differential equation. One of these is known as the Bessel function of the first kind and is noted as $J_n(\rho)$. Notice that there are two variables, n and ρ. $J_n(\rho)$ is given by the summation

$$J_n(\rho) = \sum_{l=0}^{l=\infty} \frac{(-1)^l \rho^{n+2l}}{l!\,(n+l)!\,2^{n+2l}} \tag{7·103}$$

This solution is finite at ρ equal to zero. The other solution is known as the Bessel function of the second kind and is noted as $Y_n(\rho)$ or $N_n(\rho)$, depending on the text being referred to. It is given by

$$Y_n(\rho) = \sum_{l=0}^{l=\infty} \frac{(-1)^l \rho^{-n+2l}}{l!\,(-n+l)!\,2^{-n+2l}} \tag{7·104}$$

(In both these equations the exclamation mark is a factorial symbol.) Linear combinations of the two solutions are sometimes used and they are known as Bessel functions of the third kind, or sometimes as Hankel functions.

Of the three types of functions, the equations for the cylindrical wave guides oriented as in Figure 7·12 involve only Bessel functions of the first kind, namely, $J_n(\rho)$. These functions resemble the trigonometric functions in many ways; however, one exception is that they are not repetitive. The roots, the values of ρ for which the function is zero, are not necessarily evenly spaced as they are in trignometric functions; but there are an infinite number of roots for each function. The zero-order, first-order, and second-order functions of the first kind are shown plotted in Figure 7·13. They oscillate back and forth between negative and positive values with a decrease in peak amplitude as the argument ρ is increased. The zero-order function is equal to

* For a complete discussion, see McLachlan, *Bessel Functions for Engineers*, New York, Oxford University Press, 1934.

one at ρ equal to zero, whereas the others are zero at ρ equal to zero. The values of ρ at which the curves cross the zero ordinate line are noted in the figure. These are some of the roots of the functions.

Also involved in the cylindrical wave guide equations will be the derivative of the Bessel functions of the first kind with respect to the argument or a portion of the argument. It may be obtained by taking

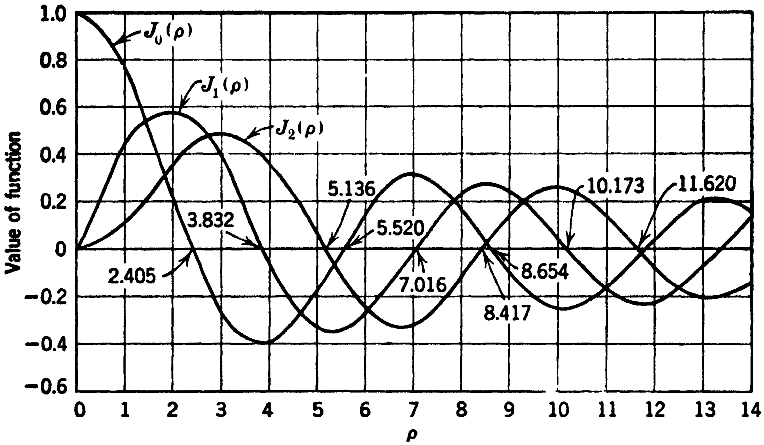


FIG. 7·13 Curves of Bessel functions of the first kind with some of the values of ρ for which they are equal to zero.

the derivative of Equation 7·103, the expansion of which yields the series for the function of the first kind. For instance, suppose it was desired to obtain the derivative of $J_n(kz)$ with respect to z. Substituting kz for ρ in Equation 7·103, we obtain

$$J_n(kz) = \sum_{l=0}^{l=\infty} \frac{(-1)^l\,(kz)^{n+2l}}{l!\,(n+l)!\,2^{n+2l}} \tag{7·105}$$

The derivative of the summation of a series is equal to the sum of the derivatives of the individual terms. Therefore, the derivative of the general term in Equation 7·105 has to be taken. It is a simple derivative to take inasmuch as it is a constant times the variable raised to the $(n + 2l)$ power. Taking the derivative of Equation 7·105, we obtain

$$\frac{\partial[J_n(kz)]}{\partial z} = \sum_{l=0}^{l=\infty} \frac{(-1)^l\,(n+2l)}{l!\,(n+l)!\,2^{n+2l}}\,(k)(kz)^{n+2l-1} \tag{7·106}$$

The right-hand member of Equation 7·106, however, can be expressed as the sum of two Bessel functions of the first kind. Adding the individual terms for n/z times $J_n(kz)$ in its expanded form to the

individual terms of $-k$ times $J_{n-1}(kz)$ will yield the right-hand side of Equation 7·106. In other words,

$$\frac{\partial[J_n(kz)]}{\partial z} = \frac{n}{z} J_n(kz) - kJ_{n+1}(kz) \qquad (7\cdot107)$$

which shows that the derivative of a Bessel function of the first kind is obtained by taking the difference between two Bessel functions of the first kind multiplied by the proper constants.

A condensed table of Bessel functions of the first kind will be found among the tables in the back of the book. This table will be found sufficient for ordinary cylindrical wave guide calculations Very often, however, it is necessary to know the roots of the first few orders of the Bessel functions of the first kind. To specify which root is meant, the first zero of the function, not counting the zero at ρ equal to zero if one occurs there, is known as rank one, the second zero as rank two, and so on. Denoting the rank as a subscript m and the root by r, r_{nm} would be the mth root, excluding zeros at ρ equal to zero, of the Bessel function of the first kind of order n. For instance, r_{12} would be the second root of $J_1(\rho)$. Referring now to Figure 7·13, we see that $J_1(\rho)$ crosses the zero ordinate at ρ equal to 3.832 first and at 7.016 second; this value of ρ equal to 7.016 is what is known as r_{12}. A few of the lower order and lower rank roots are shown in Table 7·1.

TABLE 7·1

ROOTS OF BESSEL FUNCTIONS OF THE FIRST KIND, r_{nm}

Rank (m)	$n = 0$	$n = 1$	$n = 2$
1	2.405	3.832	5.136
2	5.520	7.016	8.417
3	8.654	10.173	11.620

For some of the applications it is also desirable to know the roots of the derivative, with respect to ρ, of the Bessel functions of the first kind. They are defined in the same manner as the roots of the Bessel functions themselves but are noted as r_{nm}'; it means the root of rank m of the first derivative of $J_n(\rho)$. A short list of these roots are given in Table 7·2.

TABLE 7·2

ROOTS OF THE FIRST DERIVATIVE OF BESSEL FUNCTIONS OF THE FIRST KIND, r_{nm}'

Rank (m)	$n = 0$	$n = 1$	$n = 2$
1	3.832	1.841	3.054
2	7.016	5.331	6.705
3	10.173	8.536	9.965

7·15 TM_{nm} WAVES IN CYLINDRICAL WAVE GUIDES

As discussed in the concluding paragraphs of section 7·13, one of the types of waves possible in the cylindrical wave guide is the transverse magnetic wave wherein H_z is zero. Zero may now be substituted for H_z in Equation 7·100 and 7·101 and the resultant equations solved.* The orientation of the guide is shown in Figure 7·12; the internal radius of the guide is noted as a; the reference line op for ϕ is chosen so that the equation for E_z will include only a cosine term instead of the sum of a cosine and sine term; and the subscripts n and m again specify the specific mode being used.

The final equations for the propagation constant and for the components of the TM_{nm} wave are

$$\Gamma_{nm} = j\sqrt{\omega^2\mu_0\epsilon_0 - \left(\frac{r_{nm}}{a}\right)^2} \tag{7·108}$$

$$E_z = A[\cos n\phi]\left[J_n\left(r\frac{r_{nm}}{a}\right)\right]e^{j\omega t-\Gamma_{nm}z}$$

$$\begin{aligned} E_r = &-\Gamma_{nm}A\left(\frac{a}{r_{nm}}\right)^2[\cos n\phi]\left[\frac{n}{r}J_n\left(r\frac{r_{nm}}{a}\right)\right. \\ &\left.-\left(\frac{r_{nm}}{a}\right)J_{n+1}\left(r\frac{r_{nm}}{a}\right)\right]e^{j\omega t-\Gamma_{nm}z} \end{aligned} \tag{7·109}$$

$$E_\phi = \Gamma_{nm}A\left(\frac{a}{r_{nm}}\right)^2\frac{n}{r}[\sin n\phi]\left[J_n\left(r\frac{r_{nm}}{a}\right)\right]e^{j\omega t-\Gamma_{nm}z}$$

$$H_z = 0$$

$$H_r = -jA\omega\epsilon_0\left(\frac{a}{r_{nm}}\right)^2\frac{n}{r}[\sin n\phi]\left[J_n\left(r\frac{r_{nm}}{a}\right)\right]e^{j\omega t-\Gamma_{nm}z}$$

$$\begin{aligned} H_\phi = &-jA\omega\epsilon_0\left(\frac{a}{r_{nm}}\right)^2[\cos n\phi]\left[\frac{n}{r}J_n\left(r\frac{r_{nm}}{a}\right)\right. \\ &\left.-\frac{r_{nm}}{a}J_{n+1}\left(r\frac{r_{nm}}{a}\right)\right]e^{j\omega t-\Gamma_{nm}z} \end{aligned} \tag{7·110}$$

where m and n are integers which determine the mode being used and r_{nm} is the root of rank m of $J_n(\rho)$. Only the wave being propagated in the positive z direction is specified although the solution also includes

* See Chapter 7 of Sarbacher and Edson, *Hyper and Ultrahigh Frequency Engineering*, New York, John Wiley & Sons, 1943.

a wave being propagated in the minus z direction. Γ_{nm} is the propagation constant for the mode specified by n and m.

The propagation constant also determines a critical frequency as in the rectangular wave guides. Since this guide is also assumed to be a lossless guide, the propagation constant will either be a pure imaginary or it will introduce only attenuation and no phase shift, depending on the wave guide being used and the frequency being employed for the mode under consideration. Only attenuation will take place with no propagation when

$$\left(\frac{r_{nm}}{a}\right)^2 > \omega^2\mu_0\epsilon_0 \tag{7·111}$$

Similarly, propagation without attenuation will take place when

$$\omega^2\mu_0\epsilon_0 > \left(\frac{r_{nm}}{a}\right)^2 \tag{7·112}$$

The critical frequency is obtained by equating $\left(\frac{r_{nm}}{a}\right)^2$ to $\omega^2\mu_0\epsilon_0$ and substituting $2\pi f_c$ for ω. Solving for f_c, the critical frequency, we get

$$f_c = \frac{r_{nm}}{2\pi a\sqrt{\mu_0\epsilon_0}} \tag{7·113}$$

The phase velocity is again obtained for this type of guide by letting Γ_{nm} be equal to $j\beta_{nm}$ where β_{nm} is the phase constant. The phase velocity is equal to ω divided by the phase constant β_{nm}. Calling the phase velocity v_p, we find that

$$v_p = \frac{1}{\sqrt{\mu_0\epsilon_0 - \left(\frac{r_{nm}}{\omega a}\right)^2}} \tag{7·114}$$

The group velocity is specified by Equation 7·88, the velocity of light squared divided by the phase velocity.

7·16 THE TM_{01} WAVE IN A CYLINDRICAL GUIDE

No wave of the type where m and n both equal zero exists in cylindrical guides so that the lowest order mode is the TM_{01} mode. To obtain this mode, n is put equal to zero and m to one in the equations for the TM modes, Equations 7·108, 7·109, and 7·110. The root employed in this case is r_{01} which, from Table 7·1, is equal to 2.405. Substituting these values into the equations, we obtain

$$\Gamma_{01} = j\sqrt{\omega^2\mu_0\epsilon_0 - \left(\frac{2.405}{a}\right)^2} \tag{7·115}$$

$$E_z = AJ_0\left(r\frac{2.405}{a}\right)e^{j\omega t - \Gamma z}$$

$$E_r = \Gamma A\frac{a}{2.405}J_1\left(r\frac{2.405}{a}\right)e^{j\omega t - \Gamma z} \tag{7·116}$$

$$E_\phi = 0$$

$$H_z = 0$$

$$H_r = 0$$

$$H_\phi = jA\frac{\omega\epsilon_0}{2.405}aJ_1'\left(r\frac{2.405}{a}\right)e^{j\omega t - \Gamma z} \tag{7·117}$$

We see that both E_ϕ and H_r become identically equal to zero; hence neither of these components exist in the TM_{01} wave.

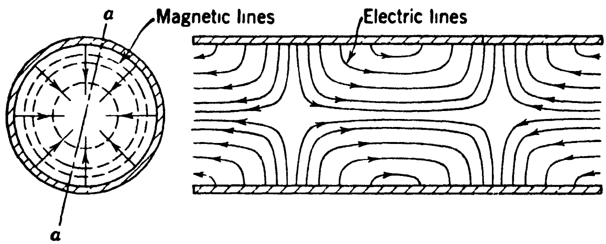


FIG. 7·14 The field patterns, at one instant of time, for the TM_{01} wave in a cylindrical wave guide.

The electric and magnetic line configurations given by Equations 7·116 and 7·117 are shown in Figure 7·14. The magnetic lines are circles concentric with the tube and perpendicular to the axis of the tube, in this case the z axis. The electric lines start on the sides of the tube, extend towards the center and along the length of the guide, re-entering the side of the tube farther down the tube. A plane passing through the axis of the guide would include the electric line loop.

With an increase in time the **E** and **H** fields travel down the length of the guide in the positive z direction with a velocity of v_p where, from Equation 7·114,

$$v_p = \frac{1}{\sqrt{\mu_0\epsilon_0 - \left(\frac{2.405}{\omega a}\right)^2}} \tag{7·118}$$

The critical frequency of this mode is obtained by substituting into Equation 7·113 the root r_{01}:

$$(f_c)_{01} = \frac{2.405}{2\pi a\sqrt{\mu_0\epsilon_0}} \tag{7·119}$$

EXAMPLE 7·4 Determine the critical frequency of an air dielectric cylindrical wave guide which has an internal radius of 2 centimeters and is to be used with a TM_{01} wave.

This problem is solved by substituting into Equation 7·119, the equation for the critical frequency.

$$(f_c)_{01} = \frac{2.405}{2\pi(0.02)} 3 \times 10^8$$

$$(f_c)_{01} = 5740 \text{ megacyles} \qquad \textit{Ans.}$$

7·17 HIGHER *TM* MODES IN CYLINDRICAL WAVE GUIDES

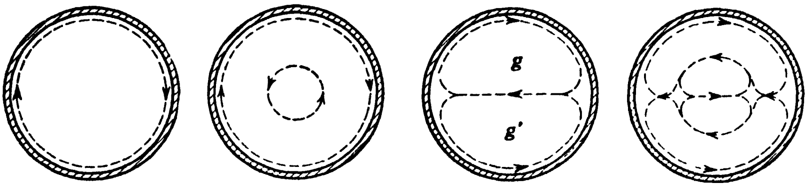
Higher modes of the *TM* waves in the cylindrical wave guide are specified by substituting larger values for m and n in Equations 7·108, 7·109, and 7·110. The resultant equations will yield the configurations of the electric and magnetic intensity lines for those modes.

The series of modes obtained with m equal to 1 and with n taking on integer values from 1 on up are very often referred to by only a single subscript. Thus the TM_{01} wave would be called the E_0 wave, the TM_{11} wave would be called the E_1 wave, the TM_{21} wave would be called the E_2 wave, and so on. All the other types are always referred to with the double subscript. However, to avoid any possible confusion, the double subscript method of notation will be used in this book.

As seen from Equation 7·110, all those *TM* modes which have n equal to zero, TM_{01}, TM_{02}, TM_{03}, and so on, will only have one component of $\mathbf{H}$, namely H_ϕ, which will not vanish. Consequently, in all these *TM* modes the magnetic lines will be concentric circles, concentric with the inside of the guide and perpendicular to the axis of the guide. This configuration is shown in the first two diagrams of Figure 7·15, (*a*) and (*b*). The dotted lines represent the peaks of the magnetic fields and the arrows indicate their direction at one instant of time. The maximums which are shown to occur next to the surface of the inside of the guide actually occur right on that surface; but, for convenience of illustration they are shown a little removed from the surface. The diagram shown in Fig. 7·15(a) can be compared to the complete configuration shown in Figure 7·14. By this comparison, what Figure 7·15(b) indicates is that the magnetic field lines are concentric circles which have zero intensity at the center and build up to a maximum part way out on the radius, decrease to zero, and build

up to a maximum, in the opposite direction of the first maximum, at the inside surface of the guide.

The TM_{11} mode is also interesting. What happens in this case is that a diametrical line is introduced. There are now two minimums, one at g and the other at g'. The magnetic lines are no longer circular but have the shape of a half circle with the corners rounded off. At



(a) TM_{01} (b) TM_{02} (c) TM_{11} (d) TM_{12}

FIG. 7·15 The maximum magnetic lines of TM waves in cylindrical wave guides. The lines occurring next to the surface of the inside of the guide should actually be right on that surface but for clarity are shown a little removed from the surface.

one instant of time the lines are clockwise around g and counterclockwise around g'. They are always in the opposite direction from one another so that at the maximum at the center they will be in the same direction, as they must be.

The TM_{12} mode is an extension of this wherein there are four minimums and the lines begin to look like packed sardines. It can be seen now that as the integer m is varied it will change the number of maximums, or closed loops, occurring along a circumferential line. As the integer n is increased, it will increase the number of maximums occurring along a radial line of the guide.

7·18 TE_{nm} WAVE IN A CYLINDRICAL WAVE GUIDE

Another type of wave which is possible in the cylindrical wave guide is the transverse electric wave. In this type of wave the lengthwise component of **E**, namely, E_z, is identically zero. Again the same method of solution is followed wherein zero is substituted for E_z in Equations 7·100 and 7·101 and the resultant equations solved. The orientation of the guide is the same as shown in Figure 7·12, and the reference line op for ϕ is again chosen to give only a cosine term for H_z. These solutions, however, involve the roots of the derivatives of the Bessel functions, r_{nm}', as given in Table 7·2.

The final equations for the propagation constant and the components of the TE_{nm} waves are as follows:

$$\Gamma_{nm} = j\sqrt{\omega^2\mu_0\epsilon_0 - \left(\frac{r_{nm}'}{a}\right)^2} \tag{7·120}$$

$$E_z = 0$$

$$\begin{aligned}
E_r &= jA\omega\mu_0 \left(\frac{a}{r_{nm}'}\right)^2 \frac{n}{r} [\sin n\phi]\, J_n\left(r\,\frac{r_{nm}'}{a}\right) e^{j\omega t - \Gamma z} \\
E_\phi &= jA\omega\mu_0 \left(\frac{a}{r_{nm}'}\right)^2 [\cos n\phi] \left[\frac{n}{r} J_n\left(r\,\frac{r_{nm}'}{a}\right)\right. \\
&\qquad \left. - \left(\frac{r_{nm}'}{a}\right) J_{n+1}\left(r\,\frac{r_{nm}'}{a}\right)\right] e^{j\omega t - \Gamma z}
\end{aligned} \tag{7·121}$$

$$\begin{aligned}
H_z &= A\,[\cos n\phi]\, J_n\left(r\,\frac{r_{nm}'}{a}\right) e^{j\omega t - \Gamma z} \\
H_r &= -\Gamma A \left(\frac{a}{r_{nm}'}\right)^2 [\cos n\phi] \left[\frac{n}{r} J_n\left(r\,\frac{r_{nm}'}{a}\right)\right. \\
&\qquad \left. - \left(\frac{r_{nm}'}{a}\right) J_{n+1}\left(r\,\frac{r_{nm}'}{a}\right)\right] e^{j\omega t - \Gamma z} \\
H_\phi &= \Gamma A \left(\frac{a}{r_{nm}'}\right)^2 \frac{n}{r} [\sin n\phi]\, J_n\left(r\,\frac{r_{nm}'}{a}\right) e^{j\omega t - \Gamma z}
\end{aligned} \tag{7·122}$$

The determining root in this case is the root of the derivative of the Bessel function of the first kind. Because Γ is similar to the propagation constant for the TM mode in a cylindrical guide except for the fact that a different root is used, the expression for the phase velocity can be obtained by substituting the root of the derivative for the root of the function in Equation 7·114:

$$v_p = \frac{1}{\sqrt{\mu_0\epsilon_0 - \left(\dfrac{r_{nm}'}{\omega a}\right)^2}} \tag{7·123}$$

The expression for the critical frequency f_c' of the TE mode can be obtained in a similar manner from Equation 7·113:

$$f_c' = \frac{r_{nm}'}{2\pi a\sqrt{\mu_0\epsilon_0}} \tag{7·124}$$

7·19 THE TE_{01} WAVE IN A CYLINDRICAL WAVE GUIDE

The equations for the TE_{01} wave in the cylindrical wave guide are obtained by substituting zero for n and one for m in Equations 7·120, 7·121, and 7·122:

$$\Gamma_{01} = j\sqrt{\omega^2\mu_0\epsilon_0 - \left(\frac{3.832}{a}\right)^2} \tag{7·125}$$

$$
\begin{aligned}
E_z &= 0 \\
E_r &= 0 \\
E_\phi &= -jA\omega\mu_0 \left(\frac{a}{3.832}\right) J_1\left(r\,\frac{3.832}{a}\right) e^{j\omega t - \Gamma_{01}z}
\end{aligned}
\tag{7·126}
$$

$$
\begin{aligned}
H_z &= AJ_0\left(r\,\frac{3.832}{a}\right) e^{j\omega t - \Gamma_{01}z} \\
H_r &= \Gamma A\left(\frac{a}{3.832}\right) J_1\left(r\,\frac{3.832}{a}\right) e^{j\omega t - \Gamma_{01}z} \\
H_\phi &= 0
\end{aligned}
\tag{7·127}
$$

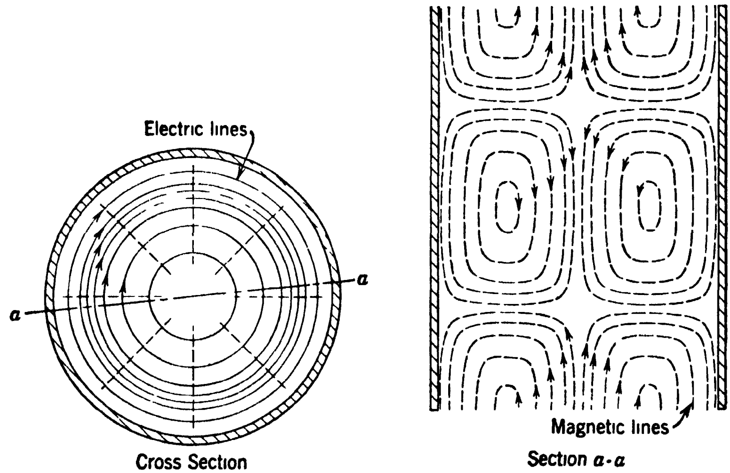


FIG. 7·16 The field patterns of the TE_{01} wave in a cylindrical wave guide at one instant of time.

We can see some resemblance between these equations and the equations for the TM_{01} wave in the cylindrical guide as given in Equations 7·115, 7·116, and 7·117; the electric components and the magnetic components have been interchanged and a different root is employed.

In Figure 7·16 is shown the configuration of the lines of force. In this case the electric lines are concentric circles, concentric with the circumference of the guide, and in a plane perpendicular to the axis of the guide. The electric field intensity, however, is a minimum at the center and at the inside surface of the guide, reaching a maximum in-between. The magnetic lines are complete closed loops that lie wholly within planes that include the axis of the guide. The magnetic field is a maximum at the center and at the inside surface of the guide.

As time increases, this configuration moves down the guide at a phase velocity determined by substituting r_{01}' into Equation 7·123:

$$v_p = \frac{1}{\sqrt{\mu_0\epsilon_0 - \left(\dfrac{3.832}{\omega a}\right)^2}} \tag{7·128}$$

The critical frequency is obtained by substituting r_{01}' into Equation 7·124:

$$(f_c')_{01} = \frac{3.832}{2\pi a\sqrt{\mu_0\epsilon_0}} \tag{7·129}$$

EXAMPLE 7·5 Determine the wavelength of a propagated TE_{01} wave inside an air dielectric cylindrical wave guide, 2 centimeters inside diameter, when the frequency being used is 25,000 megacycles.

To obtain the wavelength inside the guide, the phase velocity in the guide must be obtained first. It is obtained by substituting into Equation 7·128:

$$v_p = \frac{1}{\sqrt{0.111 \times 10^{-16} - \left(\dfrac{3.832}{(2\pi)(25 \times 10^9)(0.01)}\right)^2}}$$

$$v_p = \frac{1}{\sqrt{0.051 \times 10^{-16}}}$$

$$v_p = 4.42 \times 10^8 \text{ meters per second}$$

The wavelength, however, is equal to the velocity divided by the frequency. Dividing v_p by 25×10^9, we obtain

$$\lambda = \frac{4.42 \times 10^8}{25 \times 10^9}$$

$$\lambda = 1.77 \text{ centimeters}$$ *Ans.*

7·20 HIGHER *TE* MODES IN CYLINDRICAL GUIDES

In the same manner the higher *TE* modes can be obtained by substituting larger values of the *m* and *n* integers into the general equations for the *TE* mode, Equations 7·120, 7·121, and 7·122.

The abbreviated method of referring to the series of *TE* modes with *m* equal to 1, used for the *TM* modes, is also used. These modes can be referred to by the use of only one subscript, *n*. This means that the TE_{01} wave would be called the H_0 wave, the TE_{11} wave the H_1 wave, the TE_{21} wave the H_2 wave, and so on. All the other types are again always referred to by means of the double subscript.

As seen from Equation 7·124, the critical frequency is determined

by the value of r_{nm}'; the lower the value of r_{nm}', the lower the critical frequency. From Table 7·2 we find that the lowest value of r_{nm}' occurs when both n and m equal one. In other words, for any cylindrical wave guide the last TE mode to be cut off as the frequency is decreased would be the TE_{11} mode. This root is smaller than any of the r_{nm} roots as given in Table 7·1. Since the equations for the critical frequencies for both modes are similar, the TE_{11} mode is the last mode to be cut off as the frequency is decreased. It is this TE_{11} mode which is used when we desire to pass only one mode in a cylindrical wave guide. The cut-off or critical frequency is obtained by substituting into Equation 7·124:

$$f_{cc} = \frac{1.841}{2\pi a\sqrt{\mu_0\epsilon_0}} = \text{Lowest critical frequency for a cylindrical wave guide—}TE_{11}\text{ mode} \qquad (7\cdot130)$$

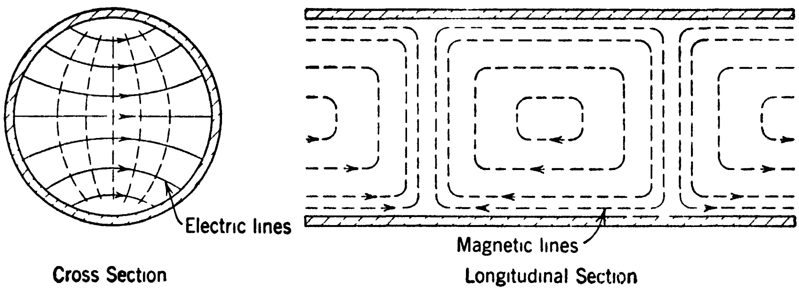


FIG. 7·17 The field patterns, at one instant of time, of the TE_{11} wave in a cylindrical wave guide.

The electric and magnetic line configurations for the TE_{11} mode are shown in Figure 7·17. We see how it resembles a TE_{01} wave in rectangular wave guides. The electric lines are lines from one side of the guide to the other and the magnetic loops are at right angles to them. Both are bent to conform to the curvature of the sides of the cylindrical guide. The maximum electric field intensity occurs across a diameter which, in the figure, is the horizontal diameter; and the maximum magnetic field intensity, in the figure, occurs at the top and bottom of the inside surface of the guide.

In Figure 7·18 are shown the lines of maximum electric field intensity for a few of the TE modes. At the surface of the inside of the guide the tangential electric field will be zero, as it must be. In (a) and (b) are shown the two modes where n is zero. The electric lines are

all circles concentric with the circumference of the guide. Changing the n to one as shown in (c) and (d) introduces a diametrical maximum which merges with the circular lines.

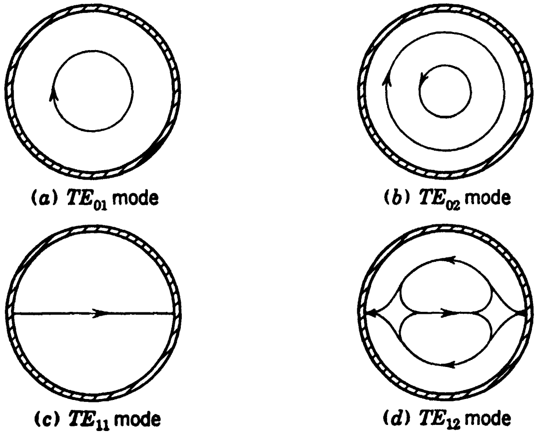


FIG. 7·18 Maximum electric intensity lines for some of the higher TE modes in cylindrical wave guides.

7·21 ATTENUATION IN WAVE GUIDES

Attenuation may take place in a wave guide when the frequency being used is below the critical frequency for the mode employed. However, in this case, no propagation takes place since no phase shift variation with length is present. This type of guide, a guide used below its critical frequency, is often employed as an attenuator, called a piston attenuator. The signal is inserted at one end of the guide and then extracted from the guide by a movable pickup. As the pickup is moved farther away from the input the signal decreases exponentially as demonstrated by the equations when Γ is real. This is interesting inasmuch as the output will be linear in decibels with the length of guide in use.

Attenuation, which is actual loss in power, also takes place in wave guides when the guide walls are not perfect conductors. In addition there may be a loss in the dielectric when some dielectric other than air is used. These losses cause the wave to attenuate even though the frequency being employed is above the critical frequency. In these cases the propagation constant, which is pure imaginary for the lossless case, will have added to it a real component, α. This α will consist of two parts, one caused by the loss in the sides of the guide and the other

by the loss in the dielectric, if such a loss exists. Equations and curves for these attenuation constants are given in the literature.*

Besides these two factors, losses introduced by joints, obstacles, or any other disturbing features will introduce attenuation. This type of attenuation is usually centered about one point in the guide and has to be considered more like a load on the guide at that point. It is only when the disturbing factor is distributed over a length of the guide that it can be considered as part of the propagation constant.

In the case of the TE_{01} mode in a rectangular wave guide there is an optimum ratio of height to width for the lowest attenuation in the guide provided the sum of height and width is kept constant. Chu and Barrow found, in their investigation of this phenomenon, that a ratio of 1.18 of the dimension of the side parallel to the **E** vector over the dimension of the other side yields the lowest attenuation.

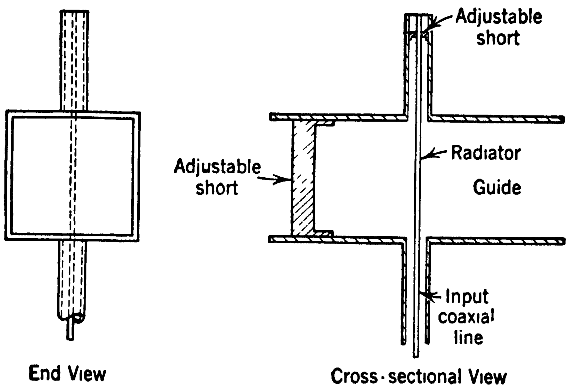


FIG. 7·19 A method of coupling to a TE_{01} wave where adjustable shorts are used for matching purposes as the frequency is varied.

7·22 COUPLING TO WAVE GUIDES

To couple energy into a wave guide some type of antenna is inserted that will radiate the energy to be transmitted. The antenna has to produce the field configuration of the desired mode. In a wave guide which passes only one mode this is not too troublesome, inasmuch as all other modes which may be excited will be quickly attenuated. However, if there is a chance for a higher mode to exist, care should

* *Microwave Transmission Design Data*, Publication 23–80, Sperry Gyroscope Co., Inc., New York.

R. I. Sarbacher and W. A. Edson, *Hyper and Ultrahigh Frequency Engineering*, New York, John Wiley & Sons, 1943.

be taken, usually in the alignment of the exciting antenna, that no higher modes are produced.

In Figure 7·19 is shown a method of coupling a TE_{01} wave into a rectangular wave guide. A coaxial line is led into the guide at the center of one side and the inner conductor, which is the antenna wire, is run across the guide. This will radiate an **E** vector which is parallel to one of the sides with its maximum at the center of the guide, exactly what is desired. To increase the radiation resistance of the wire, a coaxial loading impedance with an adjustable short is used at the terminating end of the antenna wire. This is equivalent to top loading of a vertical antenna inasmuch as it is adjusted to obtain maximum current in the radiator, the part of the wire which traverses the guide. An adjustable short is also used at the closed end of the guide so that the reflected wave from the closed end will reinforce the direct radiated wave from the wire. When only a single frequency is employed, these shorts need not be made adjustable. At a single frequency a single or double stub method of matching the input line can be employed to get the maximum power transfer into the guide.

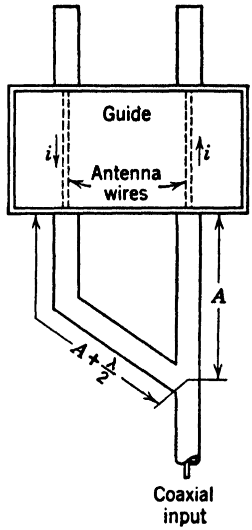


FIG. 7·20 An extension of the method of coupling shown in Figure 7·19 to a coupling for a TE_{02} wave.

For a TE_{02} wave, two antennas, similar to the one shown in Figure 7·19, are used. They are shown in Figure 7·20, where the two are placed at the maximums of the **E** vector, one-fourth and three-fourths of the way across the guide. Since they are to be excited opposite in phase, they are connected together with a half wavelength difference in line length.

The method of coupling shown in Figure 7·19 can also be used to excite a TE_{11} wave by adjusting the short on the radiating wire so that a null in current occurs at the center of the radiating wire (instead of a maximum as is used with the TE_{01} mode). Then current will flow in opposite directions on either side of center; the electric vector radiated along the top of the guide will always be 180° out of phase with the electric vector radiated along the bottom of the guide. This is what is desired, as seen in Figure 7·4.

The TM_{11} wave has magnetic lines which are closed loops perpendicular to the length of the guide and electric lines which have components along the length of the guide. This wave is generated as shown in

Figure 7·21. A coaxial line is connected to the end of the guide, at its center, and the inner conductor is projected into the guide along its axis. This arrangement will generate the desired configuration for a TM_{11} mode.

Figure 7·22 shows an extension of this method of coupling to the TM_{12} mode. In this case there are two sets of closed magnetic line

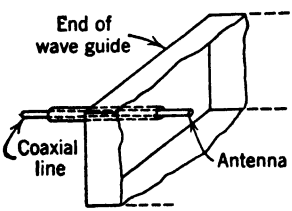


FIG. 7·21 Coupling to a TM_{11} wave in a rectangular wave guide.

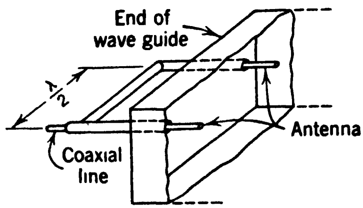


FIG. 7·22 Coupling to a TM_{12} wave in a rectangular wave guide.

loops side by side; hence two longitudinal antennas are necessary. Because the antennas have to be 180° out of phase, the two coaxial lines are connected to the input with line lengths which differ by a half wavelength.

In cylindrical wave guides the methods followed are the same. Because the configurations of the magnetic and electric lines of a TE_{11} wave in a cylindrical wave guide are similar to the configurations of the lines in a TE_{01} wave in a rectangular wave guide, the method of excitation shown in Figure 7·19 is used; except, of course, the wave guide is circular. Likewise, the line configurations for a TM_{01} wave in cylindrical wave guides are similar to the configurations for a TM_{11} wave in rectangular wave guides so that the method of coupling shown in Figure 7·23, similar to that shown in Figure 7·21, is used.

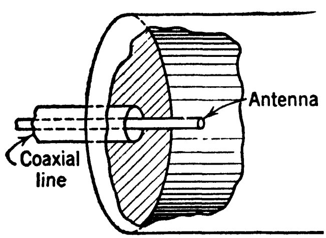


FIG. 7·23 Coupling to a TM_{01} wave in a cylindrical wave guide.

In a similar manner, any mode may be generated if the proper coupling system within the guide is used. The radiating elements have to generate the desired electric and magnetic line configuration, preferably without generating any other modes. Very often this is difficult to achieve so that wave guides which propagate only one mode, to the exclusion of all others, are often employed. Hence, in rectangular guides the TE_{01} mode is used and in circular guides the TE_{11} mode

is utilized. For instance, in the case of a rectangular guide, a guide ½ inch by 1 inch, having a critical frequency which corresponds to a wavelength of 4.57 centimeters, would be used with a TE_{01} wave between the frequencies of about 8000 to 10,000 megacycles.

The foregoing methods of coupling to a wave guide can be used for either the input or output of the guide. This follows from the conclusion that what is being used is actually a special type of confined radiation from an antenna; consequently, it will obey the reciprocity theorem discussed under antennas (section 6·14). Thus the method of coupling can be adjusted when used as either a transmitter coupling or a receiver coupling and then can be put to use as one or the other. The coupling can usually be adjusted when used as a transmitting source by employing a small pickup loop somewhere in the guide. The antenna itself, such as the loading on the end and the short in the guide, is adjusted until this pickup loop indicates a maximum in some type of indicator, usually a crystal type. The feed line is then matched by some means so that it has no standing wave on it.

7·23 Standing waves and impedances in wave guides

In ordinary transmission lines, as discussed in Chapter 1, the lines were said to be matched if there were no standing waves present on the lines. In other words, the reflected wave is zero and only one propagated wave is present, the one traveling from the source to the load. Similarly, the wave guide is said to be matched if there is no reflected wave present in the guide. This reflected wave would create a standing wave just as in the case of the ordinary transmission line, inasmuch as the transmission line is only a special case of the wave guide and has the same type of propagation constant.

The standing wave is measured by inserting a small probe, actually a small antenna, into the guide, maintaining its exact position in the cross section of the guide, and moving it back and forth along the length of the guide. In Figure 7·24 is shown a cross section of a standing-wave measuring setup, using a rectangular wave guide propagating a TE_{01} wave. A small probe is inserted into the slot which runs lengthwise along the center of the top of the guide. The slot should be as narrow as possible to avoid any leakage or coupling to the outside surface. The probe is tuned by means of a resonant coaxial line. Connected to the line is a crystal detector with a direct-current output proportional to the strength of the field within the guide. The probe should be adjusted so that the minimum insertion is used so as to create the least disturbance possible in the guide. Very heavy guide blocks are usually employed so that the position of the probe in the line will not

change with movement along the line. If it does change, the readings will be erroneous. The probe is moved back and forth along the guide, and readings are taken on the meter for various positions. When these readings are converted into proportional r-f inputs, by means of the calibration curve of the crystal, they will yield the standing wave pattern of the wave guide.

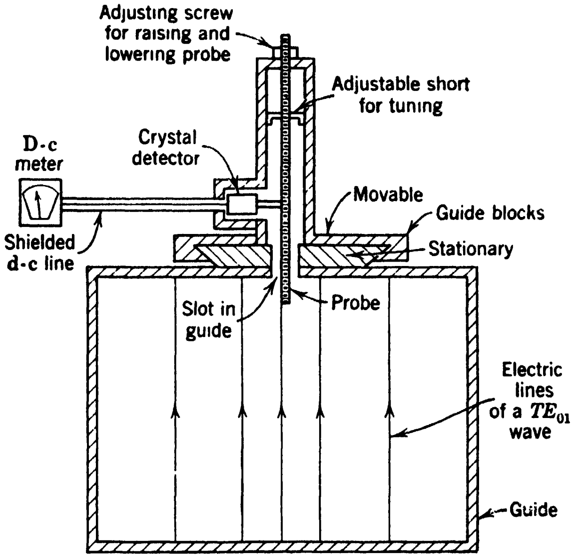


FIG. 7·24 A coaxial probe-type measuring instrument for measuring standing waves in a rectangular wave guide using the TE_{01} mode of transmission.

Another method of measuring the standing wave in the wave guide is to measure the two traveling waves separately. A device for doing this is called a unidirectional coupler, or wave selector, or reflectometer, or sometimes just "directional coupler."* It consists of a probe or combination of probes which have a directional characteristic with a null in one direction. In other words, it really constitutes an antenna array. For instance, in an end fire array of two antennas, maximum sensitivity is obtained in one direction and zero in the other.

In Figure 7·25 is shown one type of directional coupler. Two similar probes, probe I and probe II, are placed one quarter of a wavelength apart in the guide, this wavelength being measured inside the guide. These two probes are then similarly coupled to a transmission line, very

* Nathaniel I. Korman, "Note on a reflection-coefficient meter," *Proceedings of the I.R.E.*, September, 1946.

loosely so that they will not load the line or introduce an appreciable disturbance in the line. They are coupled to the line one quarter of a wavelength apart, this distance being determined by the wavelength on the transmission line. Both ends of the transmission line are matched into indicators, indicator A on one side and indicator B on the other side.

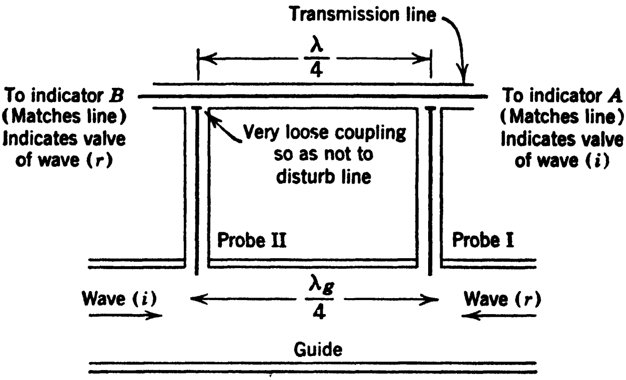


FIG. 7·25 A directional coupler employing two probes set a quarter of a wavelength apart (the wavelength being measured within the guide). The incident and reflected waves are indicated on separate meters.

Assume that the incident wave is wave i, as shown. It will introduce a current into probe I which lags the current in probe II by 90°. Both probes will introduce waves traveling in both directions on the transmission line. For simplicity, assume no reflections by either the couplings or the indicators. However, the wave generated by II traveling from II toward indicator A in the transmission line will have had a 90° delay introduced in it so that this wave and the wave traveling toward indicator A introduced by I will reinforce and indicator A will read the sum of the two waves. The wave generated by II traveling towards indicator B will have a 90° delay introduced in it so that, in combination with the condition that the current in the probe already lags 90°, it will lag the wave introduced by I, traveling toward B, by 180°. Consequently, the waves will cancel one another and indicator B will read zero for any value of wave i. Similarly, from symmetry, indicator B will read the sum of the two probe pickups for wave r, the reflected wave, and zero for wave i. Thus the magnitudes of each wave are indicated separately and the standing wave can be removed by merely watching the proper indicator and reducing its value to zero. At all times an excellent quantitative measure of the standing wave is obtained.

Sometimes, in wave guides, the measuring line is another wave guide and the couplings between it and the guide, where the waves to be measured are found, consist of openings or windows common to the two guides. Thus the two guides are placed one on top of the other and the coupling windows are nothing more than holes punched in the separating wall. Also, in other cases, instead of an end fire array, a cardioid pattern generated by a loop and wire is used. In fact, any method of obtaining a unidirectional antenna pattern offers a method of construction for a directional coupler.

In transmission line theory the characteristic impedance of the transmission line was the determining factor for the standing wave set up on the transmission line by a load impedance, Z_L. To apply transmission line theory to wave guides, some means of measuring, or method of determining a characteristic impedance of a wave guide, has to be employed. Many definitions have been proposed and the results obtained by nearly all of them are quite good. The one that seems the most popular is the specific wave impedance, the ratio of the magnitude of the transverse electric to the magnitude of the transverse magnetic fields for a given mode. For the TE_{01} mode in a dissipationless rectangular wave guide, the Z_0 of the guide is obtained by dividing E_z by H_y, neglecting the minus sign. Doing this and substituting for Γ_{01}, we obtain

$$Z_0 = \frac{\omega\mu_0}{\beta_{01}} \tag{7·131}$$

which is the characteristic impedance for an air dielectric wave guide. It may be simplified to

$$Z_0 = 4\pi v_p \times 10^{-7} \tag{7·132}$$

where v_p is the phase velocity of the wave inside the air dielectric guide. The equation for Z_0 for any mode in any air dielectric guide of uniform cross section will be found to be the same as Equation 7·132. Thus it can be used as a definition of Z_0.

If it is assumed that any higher modes generated by reflection from a load impedance or discontinuity in a guide are negligible, the reflection factor ρ may be defined as

$$\rho = \frac{\dfrac{Z_L}{Z_0} - 1}{\dfrac{Z_L}{Z_0} + 1} \tag{7·133}$$

where ρ has the same meaning as in transmission line theory. The factor ρ times the incident wave will give the reflected wave. ρ is obtained from standing wave measurements, similar to those discussed under transmission lines in section 1·14 of Chapter 1, wherein a wave guide is substituted for the standard transmission line. When obtained, ρ and the Z_0 of the wave guide being used for the measurements are then substituted into

$$Z_L = Z_0\left(\frac{1+\rho}{1-\rho}\right) \tag{7·134}$$

which is merely Equation 7·133 solved for Z_0. Thus an impedance for any discontinuity or load on a wave guide may be obtained in a manner similar to that used in transmission line measurements.

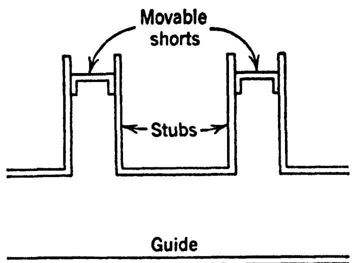


FIG. 7·26 A double stub method of matching a wave guide.

Using this concept of impedances for wave guides, we can apply the ordinary transmission line theory. Attenuation is taken into account the same way as in transmission line work. When a dielectric other than air is used the characteristic impedance is multiplied by $\sqrt{k_m/k_e}$, where k_m and k_e are the relative permeability and the relative dielectric constants of the dielectric being used.

Inasmuch as ordinary transmission line theory is applicable to wave guides, it seems reasonable to assume that a standing wave can be removed by means of stubs. This, of course, was found to be true, and both single and double stub methods have been used. In this case, however, the stubs are sections of wave guides employing shorting plates. In Figure 7·26 is shown a double stub matching arrangement. It consists of two wave guide sections attached to the main guide at two separated points. The shorting sections of the stubs are adjusted until the standing wave in the guide is removed. In a TE_{01} wave in a rectangular guide these two stubs would act like shorted transmission line stubs if they were placed in the wall parallel to the electric vectors.

Many discontinuities or configurations of stubs can be used as either series impedances or parallel impedances in guides. For instance, in a TE_{01} wave in a rectangular guide, a partition with a slit in it when placed across the guide with the slit parallel to the electric vector acts like a shunting inductance; when placed with the slit at right angles

to the electric vector it acts like a shunting capacitance. Windows in partitions, trimming screws, and many other types of discontinuities are used to advantage.

Even though standing waves can be removed from wave guide circuits by standard methods, it is advisable to avoid introducing them as a preventive measure; otherwise, the circuits become much too cumbersome to use. One difficulty is encountered at the joint where two sections of the wave guide come together. This was first used with two flanges on the guides, perfectly aligned and drawn up tight. However, any misalignment or crack in the joint would introduce reflections and leakage. One method of getting around this is the

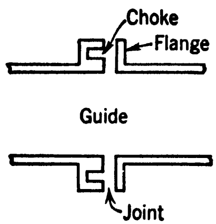


FIG. 7·27 Choke flange junction for use in wave guide assemblies.

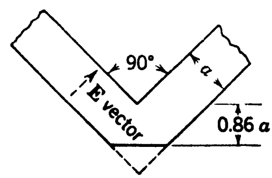


FIG. 7·28 A right angle bend in a TE_{01} mode rectangular guide propagation which does not introduce any reflections.

employment of a slot in the flange, as shown in Figure 7·27. The size and the construction of the slot were such that even though the two flanges did not touch they would offer an infinite impedance to any wave trying to get out. This type of joint is called a choke junction.

Another source of reflections in guides is a bend. If the bend is gradual it will not introduce any reflections, but if it is sharp it usually will introduce reflections. There are many ways of getting around this trouble. One method, for a right angle bend in a TE_{01} mode rectangular wave guide, is shown in Figure 7·28. This is a sketch of the inside surface of the guide. If the bend is in the plane of the electric vector, the sharp corner is replaced by a flat plate placed 0.86 times the width of the guide away from the inside corner. Doing this eliminates any reflected wave from the bend. A very complete set of curves for different types of bends, junctions, and other wave guide auxiliary apparatus may be found in the literature.*

* *Microwave Transmission Design Data,* Publication 23–80, Sperry Gyroscope Co., Inc., New York.

EXAMPLE 7·6 A slotted rectangular guide, 2 centimeters by 4 centimeters, is used to measure the equivalent impedance of a horn at 5000 megacycles which it is feeding with a TE_{01} wave. The guide is first shorted at the feed point and the positions of all the minimums are noted. The horn is then put on and a standing wave with a 1.4 ratio of maximum to minimum is measured. The distance the minimum shifted from where it was with the line shorted is 0.150 wavelengths toward the load. Determine the value of the impedance.

This problem is solved like example 1·10 in Chapter 1. The chart of Figure 1·20 can be used by drawing a line from the center to 0.150 on the wavelengths towards the load scale. This will intersect the red 1.4 circle at R/Z_0 equal to 1.05 and $+jX/Z_0$ at 0.34. The load impedance Z_L is then given by

$$Z_L = 1.05Z_0 + j0.34Z_0$$

Z_0 has to be determined. Using Equation 7·132, where v_p is given by Equation 7·89, we obtain

$$Z_0 = 4\pi \frac{10^{-7}}{\sqrt{\mu_0\epsilon_0 - \left(\dfrac{\pi}{\omega y_0}\right)^2}}$$

$$Z_0 = 4\pi \frac{10^{-7}}{\sqrt{0.111 \times 10^{-16} - \left(\dfrac{\pi}{2\pi(5 \times 10^9)(0.04)}\right)}}$$

$$Z_0 = 570 \text{ ohms}$$

Substituting this value of Z_0 into the equation for Z_L, we get

$$Z_L = 615 + j194 \qquad \textit{Ans.}$$

7·24 COMPARISON BETWEEN COAXIAL LINES AND WAVE GUIDES

Very often a decision has to be made as to whether to use a coaxial transmission line or a wave guide. Of course, for propagation at comparatively low frequencies, a wave guide is much too large. For instance, at 100 megacycles a rectangular wave guide for the TE_{01} mode would have to have for one of its cross-sectional dimensions at least 1.5 meters. Coaxial lines, on the other hand, have no low frequency cut-off so that the smallest line made will propagate any frequency; the size is chosen in accordance with the power to be carried. At very high frequencies the maximum size of the coaxial line is limited by the frequency being used. If the dimension between the inner and outer conductor is large, the coaxial line acts very much like a rectangular wave guide bent around in a circle so that the two sides come together. In other words, higher modes of

transverse magnetic and transverse electric waves may be set up in the line besides its fundamental wave, the transverse electromagnetic wave. The radial distance between the inner and the outer conductor should be kept well under a half wavelength at the frequency being used.

The power loss in the conductor is usually dependent on the surface area of the guide or line being used. Since the surface area of coaxial lines on which the current is flowing is usually much smaller, losses in the conductor are usually higher. Another important advantage of the wave guide is that no insulators or spacers are used; hence there is no loss caused by currents flowing in insulator supports.

The fact that no insulator spacers are necessary also contributes to the advantages of the wave guide in other ways. If separate spacers, placed at intervals along the line, are employed, each spacer will introduce a small reflection. At certain frequencies these reflections may add to one another, introducing a large standing wave ratio. Of course, in solid dielectric coaxial lines the center conductor is supported by a dielectric that fills the entire intervening space so that no discontinuities are present. However, the absence of any dielectric other than air, or an inert gas, in a wave guide means that the dielectric breakdowns will usually occur at much higher voltages; and if a breakdown does occur, there is nothing to carbonize or be injured that would destroy the usefulness of the line. In wave guides the concentrations of charge are usually kept farther apart, increasing the breakdown voltage.

If more than one mode is transmitted in a wave guide, the power loss goes up. Hence, for efficient operation, only one mode should be excited and all other modes that may be set up by couplings or reflections should be attenuated. Consequently, the guide should be used for only its lowest mode, sometimes called the dominant mode. A rectangular guide should be used only for the TE_{01} mode and a circular guide only for the TE_{11} mode. The frequency range of a guide is then limited by the cut-off frequency for the dominant mode on the low end and the cut-off frequency of the next higher mode on the high end. A circular guide has a frequency range of about f to $1.25f$ and a rectangular guide of f to less than $2f$, depending on the smaller cross-sectional dimension which is usually kept well under a half wavelength. A coaxial line, on the other hand, can usually be used over a very wide frequency range, there being no low frequency cut-off; a coaxial line can even carry direct current. Where very wide frequency ranges have to be covered, coaxial lines should be used or, if wave guides are used, special precautions have to be taken to prevent the origination of higher modes.

Special requirements, such as the antenna and feed problems, usually make the choice of one or the other more desirable. It may even make the choice, when using a wave guide, of a higher mode, with its attendant disadvantages, more desirable. For instance, in a rotating or swivel joint where one part, such as an antenna, has to rotate while the guide is stationary, the TM_{01} mode in a cylindrical guide will give complete rotational symmetry and is the one usually employed. If a coaxial line is used, on the other hand, its normal operating characteristics give it complete rotational symmetry; consequently, it can also be employed in a swivel joint.

7·25 Elements of cavity resonators

At extremely high frequencies, the use of ordinary circuit elements or sections of ordinary transmission lines as elements of resonant circuits becomes impractical; they become too small to handle. It becomes desirable to use wave guides as resonant circuits. This may be understood from the impedance concept of the guide. A short, which in the case of a wave guide is a conducting plate placed across the guide, reflects the incident wave with a reflection factor for the voltage vector equal to -1. Hence, if two shorting plates are placed across a guide a half wavelength apart, the wavelength being measured within the guide, the waves will be reflected back and forth in a reinforcing manner; large standing waves will be set up. If the plates are more or less than a half wave apart, the waves will not reinforce as strongly and the standing wave will have a smaller maximum value. Thus, if a voltage is introduced into the guide, the response will be similar to an ordinary resonant circuit. It is analogous to the use of transmission lines as resonant circuits.

The wavelength, λ_g, that would be measured within the guide is obtained by dividing the phase velocity by the frequency:

$$\lambda_g = \frac{v_p}{f} \tag{7·135}$$

The guide will resonate at multiples of half this value. In other words, a guide which is shorted at both ends and is an integral number of half wavelengths long will resonate. Because it is a complete enclosure on all sides it is referred to as a resonant cavity.

Cavities of many other shapes—spheres, dimpled spheres, ellipsoids, and many others—can be used as resonant cavities. The resonant frequencies of these shapes have to be obtained by the application of Maxwell's equations. The solution has to satisfy the boundary conditions of the particular cavity involved. Approximate methods

are sometimes employed where the analytical method is not yet obtainable. Before using any equations applicable to cavity resonators, we should take care to know the limitations of the equations.

The Q of a resonant cavity is a little difficult to define. In ordinary circuits the Q is defined as the inductive reactance over the series resistance. It also may be defined as 2π times the energy stored over the energy lost per cycle. The latter definition usually defines the Q of a cavity.

Cavities are used as wavemeters by coupling very loosely to the source and noting when the voltage values within the cavity are at a maximum. They are also employed as filters, selective circuit elements, and other similar devices where a resonant circuit would be used at ordinary frequencies.

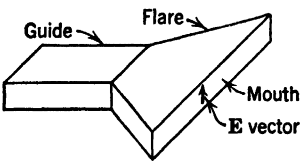


FIG. 7·29 A horn flared in the direction perpendicular to the electric vector only, for use with a rectangular guide conducting a TE_{01} wave.

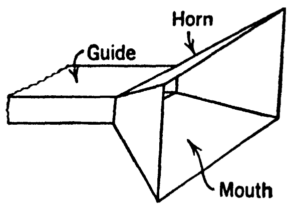


FIG. 7·30 A horn that is flared in both directions for use with a rectangular guide.

7·26 HORNS

The electromagnetic horn is used as a source of radiation of electromagnetic energy in space, very similar to the use of a horn for ordinary sound radiation. It is possible to use the horn to produce a very narrow beam for special point-to-point work or a thin plane radiation pattern for broadcast purposes.*

The horn consists of a guide which has its sides flared. In Figure 7·29 is shown a rectangular guide conducting a TE_{01} wave, terminating in a horn. The horn is flared only in the direction perpendicular to the electric vector. This type of horn yields a narrow beam in the

* W. L. Barrow and L. J. Chu, "Theory of the electromagnetic horn," *Proceedings of the I.R.E.*, Vol. 27, No. 1, January, 1939.

W. L. Barrow and F. D. Lewis, "The sectoral electromagnetic horn," *Proceedings of the I.R.E.*, Vol. 27, No. 1, January, 1939.

G. C. Southworth and A. P. King, "Metal horns as directive receivers," *Proceedings of the I.R.E.*, Vol. 27, No. 2, February, 1939.

plane of the flare and a wide beam in the plane at right angles to it. The horn in Figure 7·30, on the other hand, is flared in both directions so that the radiation beam is a slender, pencil-like result. The shape of the beam in any of the planes is dependent on the width of the mouth in that plane and the mode of the wave being fed into it.

In Figure 7·31 is shown the section of a horn consisting of two metal cones placed one above the other. This type of horn may be fed by a guide coming through the lower cone axis and feeding into the apex of the top cone or it may be fed by a coaxial line as shown dotted in

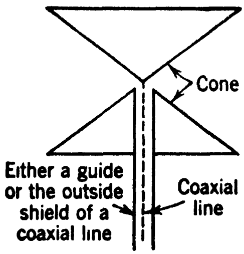
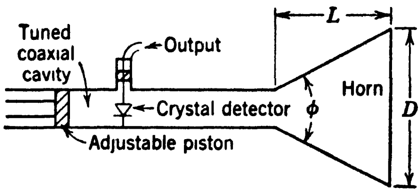


FIG. 7·31 A horn consisting of two vertical cones and used for obtaining an omnidirectional horizontal pattern. It can be fed by either a wave guide or a coaxial line as shown dotted.



Southworth and King, courtesy of the I.R.E.

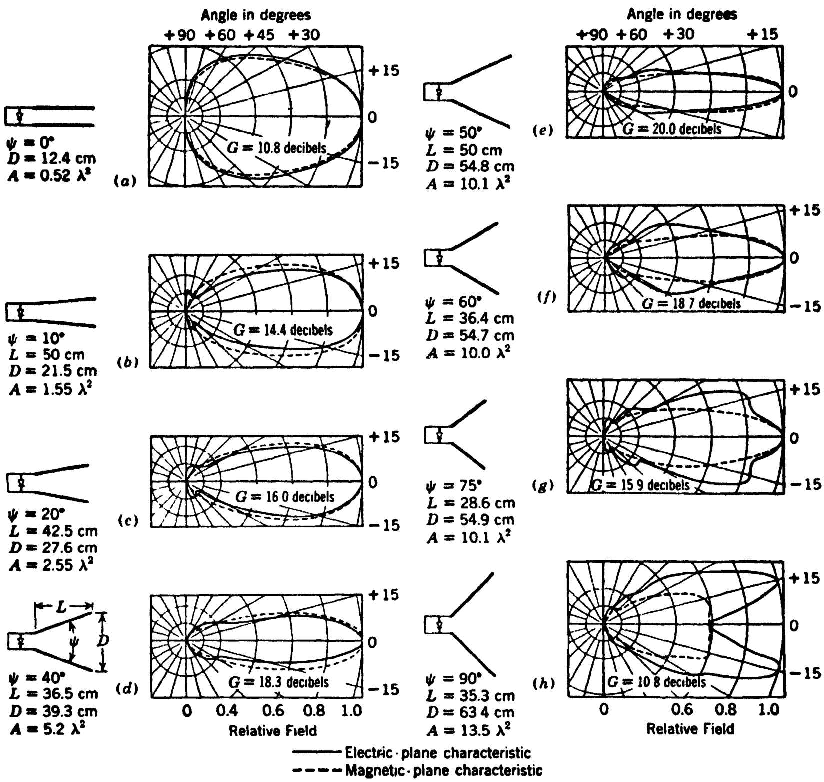
FIG. 7·32 An experimental horn used as a receiver to determine its radiation patterns.

the figure. When this horn is placed with the axis of the cones vertical, it is completely symmetrical in a horizontal plane. Hence, if it is properly fed with a symmetrical wave, such as a TE_{01} wave in a cylindrical guide, it will have a radiation pattern that is a narrow beam in the vertical plane and an omnidirectional circular pattern in the horizontal plane. The flares in these horns, which with slight modifications can be used either with circular or rectangular guides, can be straight, exponential, or any shape found desirable.

Southworth and King* have made a complete investigation of horns of circular cross section, fed with a cylindrical wave guide. Actually the tests were made on horns used as receivers, as shown in Figure 7·32. Because of the reciprocity theorem, these patterns will be the same as

* G. C. Southworth and A. P. King, "Metal horns as directive receivers," *Proceedings of the I.R.E.*, Vol. 27, No. 2, February, 1939.

the patterns obtained for the horns used as transmitters. A crystal detector was used with a coaxial line to tune it. The termination of the guide was a movable piston, also for tuning purposes. The length of the horn along its axis from the end of the guide to the mouth plane is called L, the angle of the flared sides when measured in a plane that



Southworth and King, courtesy of the I.R.E.

Fig. 7·33 The radiation patterns for circular horns of various flare angles when used at the end of a cylindrical wave guide. Measurements were taken at a wavelength of 15.3 centimeters.

contains the axis of the horn is called ϕ, and the diameter of the mouth of the horn is called D. The TE_{11} wave was used throughout so that two types of patterns are obtainable, one taken in the electric vector plane and the other taken in the magnetic vector plane. In the accompanying figures the electric plane characteristic is shown by a solid line and the magnetic plane characteristic by a dotted line.

In Figure 7·33 are shown the patterns obtainable from a series of horns of approximately the same length and varying flare angles. Notice how the best pattern is obtained at about approximately 50°.

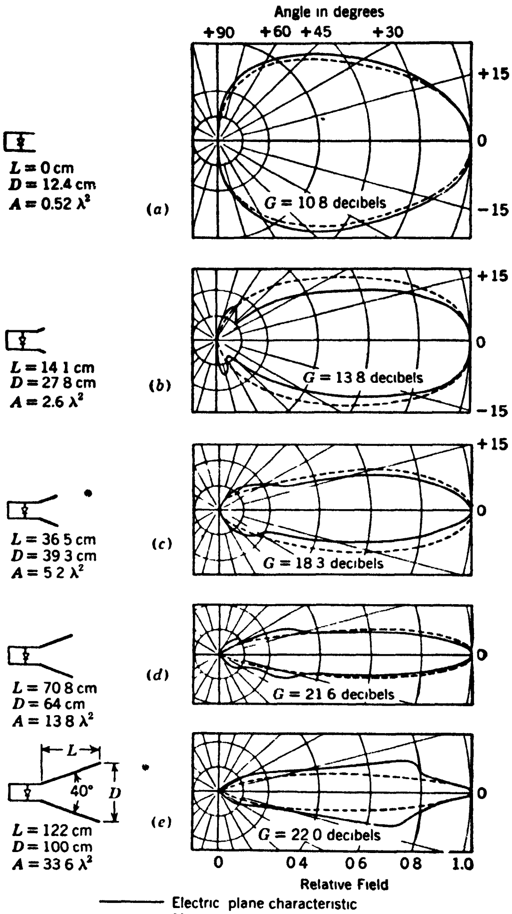


FIG. 7·34 The radiation patterns for circular horns of constant flare angle (40°) and of varying length when used at the end of a cylindrical wave guide. Measurements were made at a wavelength of 15.3 centimeters.

At that flare angle the magnetic plane characteristic and the electric plane characteristic are approximately the same, differing only by a small amount. In Figure 7·34 are shown a number of patterns obtained with the flare angle of 40° and with varying length. Notice how the patterns become progressively sharper as the length is increased.

The horns may be used as separate units or they may be used in groups of multiple units to obtain sharper patterns or specially shaped patterns.* Each of the horns in a multiple arrangement has to have an amplitude and phase control in order to control the radiation characteristics. It was found that the adjacent horns do not interact with one another to any appreciable extent.

REFERENCE READING

R. I. SARBACHER and W. A. EDSON, *Hyper and Ultrahigh Frequency Engineering*, New York, John Wiley & Sons, 1943, Chapters 5, 6, 7, 8, and 10.

R. W. P. KING, H. R. MIMO, and A. H. WING, *Transmission Lines, Antennas, and Wave Guides*, New York, McGraw-Hill Book Co., 1945, Chapter III.

S. A. SCHELKUNOFF, *Electromagnetic Waves*, New York, D. Van Nostrand Co., 1943, Chapters 8, 10, and 12.

H. H. SKILLING, *Fundamentals of Electric Waves*, New York, John Wiley & Sons, 1942, Chapter XIII.

S. RAMO amd J. R. WHINNERY, *Fields and Waves in Modern Radio*, New York, John Wiley & Sons, 1944, Chapters 8, 9, and 10.

PROBLEMS

7·1 What are the cut-off frequencies for a TE_{01} wave in a rectangular wave guide whose cross-sectional measurements are 6 centimeters by 4 centimeters?

7·2 What is the velocity of propagation, the phase velocity, of a TE_{01} wave at 6000 megacycles in the wave guide of problem 7·1? Determine the result for both methods of excitation.

7·3 Determine all the modes of the waves at 10,000 megacycles that can be propagated in a rectangular wave guide with cross-sectional dimensions 10 centimeters by 14 centimeters.

7·4 Calculate the attenuation per centimeter, for a perfectly conducting square wave guide, to a TE_{01} wave at 500 megacycles if the cut-off frequency of the wave guide for this mode is 1000 megacycles.

7·5 Determine the wavelength of the TE_{01} waves inside the guide of problem 7·2.

7·6 Determine the frequency range over which it is best to limit frequencies for a 3 centimeter diameter cylindrical wave guide.

7·7 A 4 centimeter by 2 centimeter rectangular guide is used as a slotted guide to measure the equivalent impedance of a horn. A standing wave ratio of 1.8 is measured; the minimum of the standing wave is 0.08 wavelength closer to the generator from where it was with the guide shorted at the input to the horn. What is the equivalent impedance?

* W. L. Barrow and Carl Shulman, "Multiunit electromagnetic horns," *Proceedings of the I.R.E.*, Vol. 28, No. 3, March, 1940.

Chapter 8

COMPLEX TRANSMISSION LINE NETWORK ANALYSIS

8·1 Introduction

This chapter outlines a method of reducing a so-called complex transmission line network to a conventional transmission line circuit, permitting the application of ordinary equations. A complex transmission line circuit is one which cannot be recognized easily as a network of impedances; and the problem is to find the equivalent circuit which will allow the necessary calculations to be made. The method employed* uses a number of theorems concerning the currents in coaxial and balanced transmission lines and on their shields. From these theorems the currents flowing in the lines and shields are determined and the equivalent circuit obtained.

8·2 Grounds at ultrahigh frequency

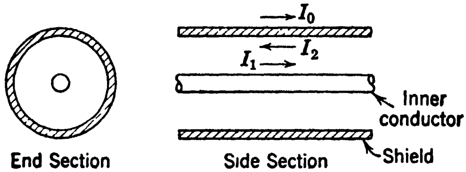
It must be emphasized that there is no such a thing as a practical common ground at ultrahigh frequency. A ground in the lower frequency range is a line, a surface, or a volume, all points of which always remain at exactly the same potential and phase. This is not feasible when the dimensions of the line, surface, or volume approach a large fraction of a wavelength A bus bar in the broadcast band, for example in a broadcast receiver, can be considered a ground line; but at 100 megacycles it would be a transmission line and would no longer have the same potential at all points.

At ultrahigh frequencies, instead of grounds, consideration of the equilibrium conditions that must be maintained on the transmission lines used in the circuits is required. Another important factor, as will be evident from the discussion, is the continuity of shield that must be maintained at all times to avoid coupling to extraneous currents. It is this continuity of shield that also prevents coupling through radiation which is very often present at these high frequencies.

* N. Marchand, "Complex transmission line network analysis," *Electrical Communication*, Vol. 22, No. 2, 1944.

8·3 Coaxial transmission line equilibrium conditions

The coaxial transmission line consists of a conducting wire concentrically disposed in a hollow conducting tube, the intervening space being filled with a dielectric which may be air. In Figure 8·1 are shown two sectional views of a coaxial transmission line with three currents, I_1, I_2, and I_0, indicated in the side section. When a transmission line of this type has a shield which is well constructed and is used above



Courtesy of *Electrical Communications*

Fig. 8·1. The currents flowing in a coaxial transmission line.

about 50 megacycles, it can be assumed, without loss of generality, that the shield is perfect. This does not imply that the shield is a perfect conductor but rather that there is no coupling between I_0, the current on the outside surface of the shield, and I_1, the current on the center conductor, or I_2, the current on the inside surface of the shield. Consequently, the current on the inside of the shield, I_2, must be equal and opposite to the current, I_1, on the center conductor. The problem of maintaining proper equilibrium conditions in the coaxial line is to prevent any coupling between the true currents, I_1 and I_2, and the interfering or unbalancing current, I_0, that might be induced on the outside of the shield. It is also important, from power considerations, to maintain constant characteristic impedance along the line.

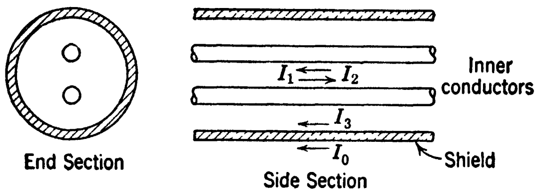
The first current theorem may now be stated:

A. The current on the inside surface of the shield of a well-shielded coaxial transmission line is equal and opposite to the current on the inner conductor when both currents are measured at the same cross-sectional plane of the line.

8·4 Balanced transmission line equilibrium conditions

The balanced transmission line consists of two parallel conductors in a dielectric which may or may not be surrounded by a symmetrically disposed shield. For shielding purposes, all balanced lines should, if possible, be enclosed in a shield. Figure 8·2 illustrates an end and side sectional view of such a line. In the diagram are shown four currents. I_1 and I_2 are the currents flowing on each of the two inner

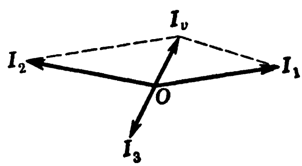
conductors; I_3 is the total resultant current flowing on the inside surface of the shield; and I_0 is the current flowing on the outside surface of the shield. In a perfectly balanced line, I_1 would be equal and opposite to I_2, and I_3 would be zero. If I_1 is not equal and opposite to I_2, then I_3 will be equal and opposite to the vector sum of I_1 and I_2, all of the currents being measured at the same cross-sectional plane of the line.



Courtesy of *Electrical Communications*

FIG. 8·2 The currents flowing in a balanced transmission line.

This relationship between the currents is shown in Figure 8·3, where I_1 and I_2 are not exactly 180° out of phase. Their sum is no longer zero but equal to the so-called unbalanced current, I_v, flowing along the transmission lines. This current, I_v, behaves as though it were a current flowing in the same direction along both the transmission lines in parallel. The lines may be considered to be acting jointly as the inner conductor of a coaxial line with the return current flowing on the inside of the shield. This return current would be equal to I_3. I_0 is again the extraneous current induced on the outside of the shield.



Courtesy of *Electrical Communications*

FIG. 8·3 A phasor diagram of the currents flowing in a balanced transmission.

The first problem encountered in maintaining the normal equilibrium conditions in a balanced transmission line is to keep I_1 equal and opposite to I_2 so that I_3 will be zero; and the second problem is to prevent I_0 from coupling into the transmission line at any point. Again, from power considerations, it is important to maintain the characteristic impedance along the line constant.

From these considerations the second theorem is obtained:

B. The current on the inside surface of the shield of a well-shielded balanced transmission line is equal and opposite to the vector sum of the currents on the two inner conductors, all of the currents being measured at the same cross-sectional plane of the line.

8·5 Continuity of currents

In a transmission line network there can be no accumulation of current at any point in the circuit provided that the point is considered to have no physical dimensions. A cross-sectional plane is usually considered electrically similar to a point for purposes of analysis of currents flowing on the inside and outside surfaces of shields. This consideration leads to the following theorem:

C. At any point in a transmission line network, the sum of all the currents flowing in and out of that point is zero.

This is an extremely valuable theorem in analyzing transmission line networks since it is now possible to determine whether or not an impedance has been introduced into the circuit at any point. Whether the current is a series current or a parallel branch current determines whether a series or a parallel impedance has been introduced.

From the impedance concept in transmission line circuits another theorem is obtained:

D. When there are two equal and opposite currents flowing in and out of a passive transmission line network, it can be replaced by an impedance, the magnitude and phase of which may be a function of frequency.

8·6 Method of analysis

Gathering together the four theorems that have been formulated, we have:

A. The current on the inside surface of the shield of a well-shielded coaxial transmission line is equal and opposite to the current on the inner conductor, when both currents are measured at the same cross-sectional plane.

B. The current on the inside surface of the shield of a well-shielded balanced transmission line is equal and opposite to the vector sum of the currents on the two inner conductors, all the currents being measured at the same cross-sectional plane of the line.

C. At any point in a transmission line network, the sum of all the currents flowing in and out of that point is zero.

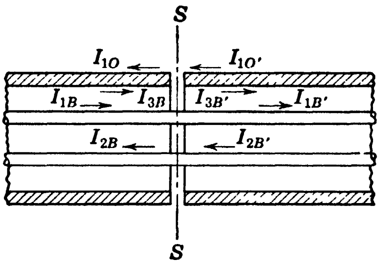
D. If there are two equal and opposite currents flowing in and out of a passive transmission line network, it can be replaced by an impedance, the magnitude and phase of which may be a function of frequency.

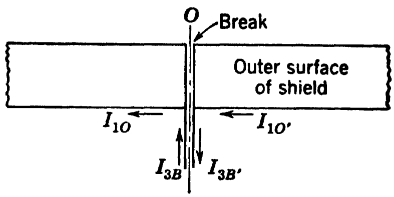
From these theorems and a schematic diagram of the complex transmission line network the analysis may be undertaken. All the currents at the discontinuities and junctions of the network are labeled. Next as many relationships between the currents as are possible from the above theorems are written down. This usually leads to an equivalent circuit that meets the requirements of the current relationships noted.

If it is found that the circuit tried does not meet the current relationships noted, the two circuits are not equivalent. The method will become more evident from the applications which follow.

8·7 Break in the shield of a balanced line

Let us consider a break in the continuity of shield of a balanced line as shown at S in Figure 8·4. The currents I_{1B} and I_{2B} are the currents in the lines and I_{3B} is the shield current representing the unbalanced current. The capacitance across the break is assumed to be negligible. The unprimed subscripts represent the values on one side of the break and the primed subscripts represent values on the other side of the break in the shield. The thickness of the shield





Courtesy of *Electrical Communications*

Fig. 8·4 The existing currents at the break in the shield of a balanced transmission line.

Fig. 8·5 The equivalent circuit for the shield of Figure 8·4.

is assumed negligible with respect to a wavelength. If the plane at which the currents are compared is passed through the break and the width of the break is assumed to be negligible in size but not zero, the following current relationships may be obtained from the theorems:

$$\begin{aligned} -I_{1B} &= I_{1B'} \\ -I_{2B} &= I_{2B'} \\ -I_{3B} &= I_{1O} \\ -I_{3B'} &= I_{1O'} \end{aligned} \tag{8·1}$$

where I_{1O} represents the current flowing on the outside of the shield. Since the unbalanced current I_{3B} depends on the vector sum of I_{1B} and $I_{2B'}$ and $I_{3B'}$ depends on the vector sum of $I_{1B'}$ and $I_{2B'}$,

$$-I_{3B} = I_{3B'} \tag{8·2}$$

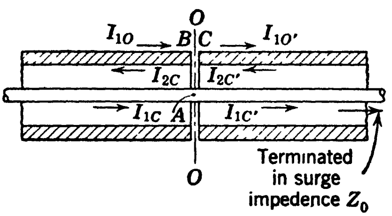
Thus, from current considerations, we see that the currents on the lines continue straight through but that the unbalanced current flowing on the inside of the shield is directly connected to the outside surface of the shield. In Figure 8·5 is shown the equivalent circuit for this

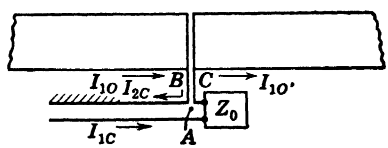
case. It is really an antenna, or other circuit, made up of the outside of the shield and being fed at the break by the unbalanced current. This result leads to the conclusion that if the unbalanced current, I_{3B}, can be reduced to zero, the break in the shield will not cause any coupling between the inside currents and the external currents. This is true inasmuch as when I_{3B} is zero, I_{1O} is zero.

8·8 Break in the shield of a coaxial line

In Figure 8·6 is shown a section of a coaxial line with a break in the shield at O. The break is again assumed to be very small in size but not zero, and the capacitance, across the break, negligible. In this case I_{1C} and I_{2C} are the currents on the inner conductor and on the shield inside surface and I_{1O} represents the current flowing on the outside surface of the shield. Again the unprimed subscripts represent the currents on one side of the break and the primed subscripts the currents on the other side. From the current theorems:

$$
\begin{aligned}
I_{1C} &= -I_{1C'} \\
I_{1C} &= -I_{2C} \\
I_{1C'} &= -I_{2C'} \\
I_{2C} &= -I_{1O} \\
I_{2C'} &= -I_{1O'} \\
I_{1O} &= -I_{1O'}
\end{aligned}
\qquad (8·3)
$$





Courtesy of *Electrical Communications*

Fig. 8·6 The currents flowing when there is a break in the shield of a coaxial line.

Fig. 8·7 The equivalent circuit for the break in a coaxial line shield (Figure 8·6).

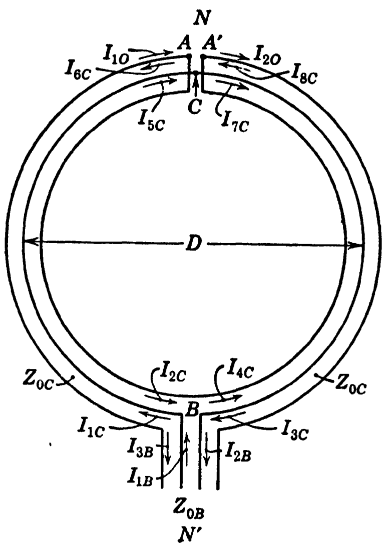
The last equation follows from the first five. The first three equations show that I_{2C} is equal and opposite to $I_{2C'}$. This result, in combination with the equalities of the fourth and fifth equations, yields the last equation. Between the points B and C the equal and opposite currents, I_{1O} and $I_{1O'}$ can be considered as an effective two-terminal impedance. The equal and opposite currents, $I_{1C'}$ and $I_{2C'}$ can also be considered as an effective two-terminal impedance which, if the line is terminated

in the characteristic impedance Z_0, will be equal to Z_0. This impedance will be across A and C.

The equivalent circuit is shown in Figure 8·7. Thus the break in the shield introduces a series impedance, which may be radiating, made up of the outside of the shield and being fed at the break by the coaxial current itself. A break in the shield of a coaxial line always introduces unity coupling between the outside currents in the shield and the coaxial line currents.

8·9 Analysis of the Shielded Loop

In Figure 8·8 is shown a section of a shielded loop, of mean diameter D, consisting of a wire inside a shield with the shield broken at the top of the loop, at the point AA'. The distance AA' is assumed to be very small but not zero, and any capacity across AA' is assumed to be negligible. The loop is thus composed of two coaxial lines, one running between the points B and A and the other between the points B and A'. At the point B there are two currents, I_{1B} and I_{2B}, which are the currents in the balanced line entering the loop. I_{1B} continues at the point B to become I_{1C} and I_{2B} continues to become the current I_{3C}. I_{1C} will have an equal and opposite current flowing along the inside of the shield, noted as I_{2C}. I_{3C} will have an equal and opposite current flowing along its shield, noted as I_{4C}. If there is any difference between the currents I_{1C} and I_{3C}, it will result in an unbalance current, I_{3B}, flowing along the inside of the balanced line shield. For I_{3B} to be zero, it must be shown that I_{3C} is equal and opposite to I_{1C}. With the break in the loop shield located at the center diametrically opposite to B, if it is shown that the coaxial currents are equal at AA', we shall conclude that the currents I_{1C} and I_{3C} at B are equal and, hence, no unbalance current I_{3B} will flow.



Courtesy of *Electrical Communications*

FIG. 8·8 A section of a balanced loop antenna showing the currents that are flowing.

First, at A, I_{5C} flows into the break with an equal and opposite current I_{6C} flowing on the inside of the shield. However, at the point

A there is also a current I_{1O} flowing on the outside of the shield. Then, in accordance with theorem C, $-I_{1O}$ must equal I_{6C}. Inasmuch as the break is very small, I_{5C} will equal the current shown as $-I_{7C}$. (Any stray capacity coupling is neglected.) The inner shield current, in equilibrium with I_{7C}, is noted as I_{8C}, and it is observed at A' that I_{8C} must equal the external current $-I_{2O}$. Thus the following equalities apply:

$$
\begin{aligned}
I_{5C} &= -I_{7C} \\
I_{5C} &= -I_{6C} \\
I_{7C} &= -I_{8C} \\
I_{6C} &= -I_{1O} \\
I_{8C} &= -I_{2O}
\end{aligned}
\tag{8·4}
$$

Therefore,

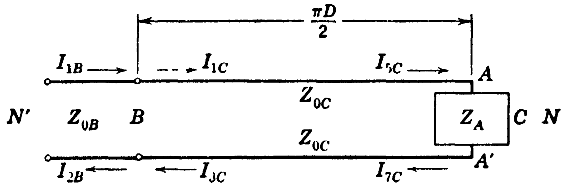
$$
I_{1O} = -I_{2O} \tag{8·5}
$$

The five equations of Equation 8·4 are obtained by the use of the theorems. The first three show that I_{6C} is equal to $-I_{8C}$. This result, in combination with the fourth and fifth equations, yields the equality shown in Equation 8·5.

Thus, as long as I_{5C} has no place to flow other than to I_{7C}, it will force a balance of I_{1O} and I_{2O}. Consequently, when this type of loop is used as a receiving antenna, if any similar parallel currents flow up the shield, such as currents that might be induced by mast effect (the mast and the two sides of the loop in parallel acting as an antenna), they will encounter a very high impedance at the break AA' and no current will flow into the coaxial lines from that source. On the other hand, the current induced in the loop acting as an antenna will be a circulating current wherein I_{1O} will be equal and opposite to I_{2O}. This induced loop current will therefore be unity coupled to the wire within the shield. In this manner when the loop is mounted in a vertical position the loop discriminates between the power picked up by the loop and the power picked up by the mast and loop acting as a vertical antenna. For a multiturn shielded loop the analysis is the same except that instead of the voltage and current being applied to only one inner conductor the same voltage acts on each turn and the same current is divided equally among the turns. Hence for an n turn loop there is a voltage gain of n-fold, the current being decreased to one n'th at the output.

To obtain the equivalent circuit of the shielded loop shown in Figure 8·8, a neutral plane NN' should be passed through the center line of the loop. This simplification can always be used with a perfectly symmetrical circuit, such as the shielded loop. Figure 8·9

illustrates the equivalent circuit which meets the requirements of Equations 8·4 and 8·5. The balanced line carrying the current I_{1B} joins the coaxial line carrying the current I_{1C} at the point B. The coaxial line then extends for a length equal to half the circumference of the balanced loop, $\pi D/2$. At the end of the coaxial line, which is at point A, the impedance Z_A is joined to the coaxial line. It is directly coupled since, as shown in Figure 8·8, I_{1C} continues directly to become I_{6C}. Thus, between the points A and C, half the impedance of the



Courtesy of *Electrical Communications*

FIG. 8·9 The equivalent circuit for the balanced loop antenna shown in Figure 8·8.

antenna exists. A similar analogy can be drawn for the other half of the loop to obtain the complete circuit.

The impedance of the loop antenna, Z_A, is the impedance across AA' of the loop measured with the inner conductor removed. This impedance is thus placed at the end of a "twin coaxial" type of balanced transmission line having a characteristic impedance of twice the characteristic impedance of each of the similar coaxial lines. The balanced connecting line is joined to this twin line at the point B.

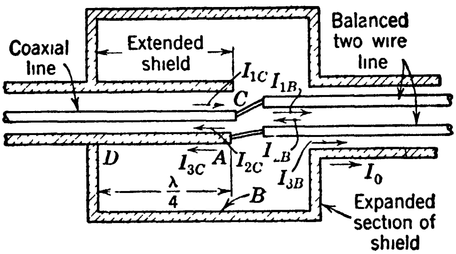
8·10 TRANSMISSION LINE CONVERSION TRANSFORMERS*

In high frequency power transmission it is frequently necessary to use a balanced two-wire line over part of the distribution system and a coaxial transmission line over the remainder of the system. The natures of the generator and the load input will dictate the particular combination of balanced two-wire and coaxial transmission lines which should be used.

At the juncture of the two types of lines, a conversion transformer is necessary to maintain the equilibrium conditions on both the lines. The conversion from one line to the other can be made ideal at only one frequency, but methods will be described which provide satisfactory conversion over a range of frequencies. The problem is to convert from a balanced transmission line to a coaxial transmission line by

* N. Marchand, "Transmission-line conversion transformers," *Electronics*, December, 1944.

introducing some type of transmission line transformer. First, to prevent the current on the outside of the shields from coupling into the lines, the outside shields of both the lines should be kept continuous so that there can be no way in which the outside current can get into the lines inside the shield. If this method is not possible, some other means must be used to prevent the current flowing on the outside of the shield from coupling into the lines. Second, to maintain I_1 equal and opposite to I_2 in the balanced transmission line, as shown in Figure 8·2, the impedances from each conductor to the shield must be maintained equal. Any difference in impedance will cause I_1 to differ from I_2. Third, the continuity of the characteristic impedance, or the impedance match, should be maintained as close as possible to obtain maximum power transfer.



Courtesy of *Electronics*

FIG. 8·10 A cross section of a single-frequency conversion section showing how the coaxial shield can be extended to reduce I_3 to zero at resonance.

8·11 SINGLE FREQUENCY TRANSFORMER

In Figure 8·10 is shown one type of transmission line transformer that can be used for single frequency transformation. A large box, an expanded section of the shield, is used. This box, or pipe, must totally enclose the interior circuits to avoid coupling between I_0, the current on the outside of the shield, and the inside currents. The coaxial line shield is extended a distance of one-quarter wavelength inside the box; and the dual line is brought in on the other side. One conductor of the dual line is connected to the inner conductor of the coaxial line and the other to the coaxial shield extension. I_{1B} and I_{2B} are the currents on the conductors of the balanced line at the junction. I_{3B} is the unbalanced current flowing on the inside of the shield. I_{1C} and I_{2C} are the equal and opposite currents in the coaxial line. I_{3C} is the current flowing on the outside of the extended shield.

To determine the equivalent circuit of this conversion transformer, the current paths have to be traced. Current I_{1B}, as shown in the

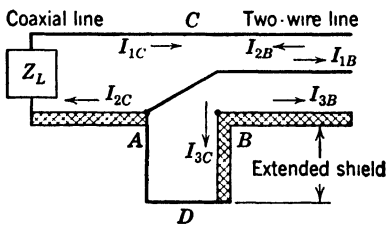
figure, will flow directly from one side of the balanced line to the inner conductor of the coaxial line, becoming I_{1C}. The current I_{2B} will flow from the other side of the balanced line and divide into two currents, I_{3C} and I_{2C}. The following equalities can be written down from the theorems:

$$\begin{aligned} I_{1C} &= -I_{2C} \\ I_{1B} &= -I_{1C} \\ I_{2B} &= -I_{2C} - I_{3C} \\ I_{3B} &= -I_{2B} + I_{1B} \end{aligned} \tag{8·6}$$

To maintain the equilibrium conditions, I_{3B} should be zero. Substituting for I_{2B} and I_{1B} in the last equation and noticing that I_{1C} is equal to I_{2C}, we obtain

$$I_{3B} = I_{3C} \tag{8·7}$$

These equalities assume that the space between the end of the extended shield and the beginning of the balanced line is negligible. Thus, if I_{3C} is zero, I_{3B} will be zero. In the transformer shown it will be zero; the outer side of the coaxial shield AD and the inner surface of the box form a coaxial transmission line shorted at the end D and a quarter wavelength long, insuring an extremely high impedance between point A and the surface at B.



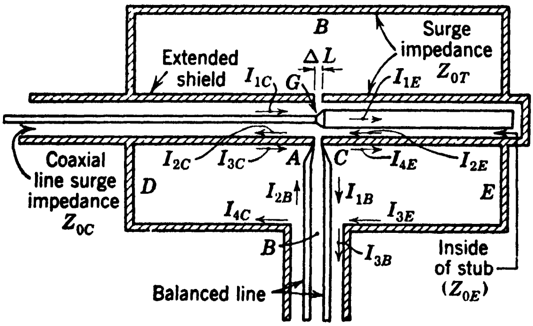
Courtesy of *Electronics*

FIG. 8·11 The equivalent circuit of the conversion transformer shown in Figure 8·10. The shield is represented by shading.

The equivalent circuit is shown in Figure 8·11. The wire representing the inside of the shield is shown shaded. It can be seen here that I_{3C} is being fed into a shorted transmission line ABD which, if it is a quarter wavelength long, will result in an infinite impedance, in the ideal case, at AB; hence I_{3C} will be zero. Whatever impedance is presented across AC, looking into the coaxial line, will be the load across the end of the balanced line. Thus, if both the coaxial line and the balanced lines have the same characteristic impedances, and if

the load impedance, Z_L, is matched to the coaxial line, the balanced two-wire line will also be matched.

The one unsatisfactory feature of this type of conversion transformer is that it yields good balance at only one frequency. As soon as the frequency is shifted, so that ADB is no longer a quarter wavelength long, I_{3C} will no longer be zero; hence I_{3B}, which is equal to I_{3C}, will also no longer be zero. Consequently, when the frequency is shifted the equilibrium conditions will no longer be obtained.



Courtesy of *Electronics*

FIG. 8·12 A wide band conversion transformer which, when the frequency is shifted, will not disturb the equilibrium conditions and will introduce only a very small mismatch.

8·12 WIDE BAND CONVERSION TRANSFORMER

To improve the transformer shown in Figure 8·10, the next logical step is to put an impedance from C to B that will be similar to the impedance at AB, so that even if the frequency is shifted it will only introduce a mismatch and will not disturb the current balance. This is true because the voltage at C is equal and opposite to the voltage at A, both with respect to B.

To make the transformer wide band without any phase shift at the junction, the transformer is constructed as shown in Figure 8·12. Instead of the second side of the balanced line being connected to the coaxial center conductor, it is connected to a shield which is a mirror image of the extended coaxial line shield. A series impedance is put into the circuit by extending the coaxial line into this new shield to yield a characteristic impedance Z_{0E}, given by

$$Z_{0E} = \frac{(Z_{0C})^2}{2Z_{0T}} \tag{8·8}$$

where Z_{0C} is the coaxial line characteristic impedance and Z_{0T} is the

characteristic impedance obtained by using the outside of the extended shield as the inner conductor and the inside of the box as the outer conductor of a coaxial line. Thus the inner conductor of the extended portion of the coaxial line must be made larger than the inner conductor of the coaxial line proper.

The lengths of the extended shields are made resonant as a quarter wavelength line at the mean frequency, f_0. If the capacitance across ΔL were neglected, the length from the break to the end of the box would be 90 electrical degrees. The extended line is left open at the end and is made resonant as a quarter wavelength open stub.

Four new currents are introduced. Two are the equal and opposite currents, I_{1E} and I_{2E}, in the open stub and the currents, I_{4E} and I_{3E}, on the outside of the shield surrounding the stub and on the inside of the box. Because of symmetry, I_{3C} and I_{4E} will be equal and opposite, yielding a balanced condition; hence no unbalanced current, I_{3B}, will flow. Writing down the current relationships from the theorems, we obtain

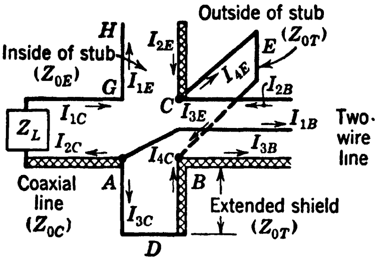
$$\begin{aligned}
I_{3C} &= -I_{4E} \\
I_{1C} &= -I_{2C} \\
I_{1E} &= -I_{2E} \\
I_{1C} &= -I_{1E} \\
I_{2B} &= -I_{2C} + I_{3C} \\
I_{1B} &= -I_{2E} + I_{4E} \\
I_{3C} &= -I_{4C} \\
I_{3E} &= -I_{4E} \\
I_{3B} &= -I_{3E} + I_{4C} \\
I_{3B} &= +I_{1B} - I_{2B}
\end{aligned} \tag{8·9}$$

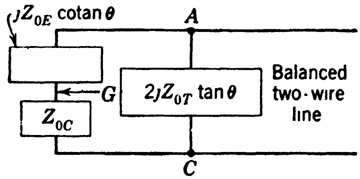
From the first four equations, we see that I_{2C} is equal and opposite to I_{2E} and, since I_{3C} and I_{4E} are equal and opposite, the equilibrium conditions will be held for all frequencies. In Figure 8·13 is shown the equivalent circuit of the transformer illustrated in Figure 8·12. This can be further modified to the circuit shown in Figure 8·14.

Z_{0C} is the characteristic impedance of the coaxial line and, when Z_L is made equal to Z_{0C}, it is the impedance that the coaxial line will present at the junction. This impedance is in series with the open-ended impedance of line GH. Across the balanced line are the two shorted sections AD and CB. In both these cases the electrical length of the lines are called θ. At f_0, θ is equal to 90 electrical degrees. The input impedance from the balanced line across AC, for a terminated

coaxial line, is given by Z, where

$$Z = \frac{(Z_{0C} - jZ_{0E} \text{ cotan } \theta)(2jZ_{0T} \tan \theta)}{Z_{0C} - jZ_{0E} \text{ cotan } \theta + 2jZ_{0T} \tan \theta} \tag{8·10}$$





Courtesy of *Electronics*

FIG. 8·13 The equivalent circuit of the transformer shown in Figure 8·12. The currents shown are the currents which are present at the junctions.

FIG. 8·14 The lumped constant equivalent circuit of the transformer shown in Figure 8·12, representing a reduction of the equivalent circuit shown in Figure 8·13.

Equation 8·10 may be simplified to

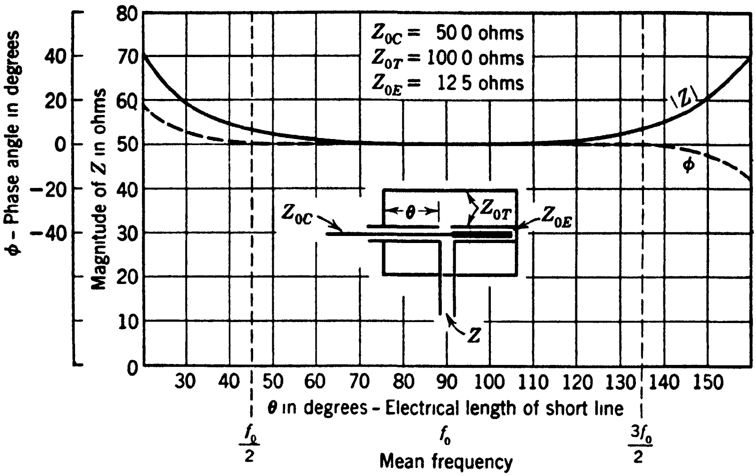
$$Z = Z_{0C} \frac{1 - j\left(\dfrac{Z_{0E}}{Z_{0C}}\right) \text{cotan } \theta}{1 - \left(\dfrac{Z_{0E}}{2Z_{0T}}\right) \text{cotan}^2 \theta - j\left(\dfrac{Z_{0C}}{2Z_{0T}}\right) \text{cotan } \theta} \tag{8·11}$$

When the value for Z_{0E} given in Equation 8·8 is substituted into Equation 8·11, it becomes

$$Z = Z_{0C} \frac{1 - j\left(\dfrac{Z_{0C}}{2Z_{0T}}\right) \text{cotan } \theta}{1 - \left[\left(\dfrac{Z_{0C}}{2Z_{0T}}\right) \text{cotan } \theta\right]^2 - j\left(\dfrac{Z_{0C}}{2Z_{0T}}\right) \text{cotan } \theta} \tag{8·12}$$

As long as Z_{0T} is kept large, this is an excellent transformer. The calculated impedance characteristic of a transformer, where Z_{0C} is equal to 50 ohms, Z_{0T} equal to 100 ohms, and Z_{0E} equal to 12.5 ohms to agree with Equation 8·8, is shown in Figure 8·15.

This type of transformer is excellent for a transfer from a stationary member to a rotating member. Since the coaxial line does not make contact with any other line, it can be kept stationary while the whole transformer and balanced line rotates around it.



Courtesy of *Electronics*

FIG. 8·15 The theoretical variation of impedance seen by the balanced two-wire line, when converted to a coaxial line by the transformer of Figure 8 12, shows but slight change in magnitude and negligible change in angle over a one-to-three change in frequency

PROBLEMS

8·1 Determine the input impedance across aa' of Figure 8·16.

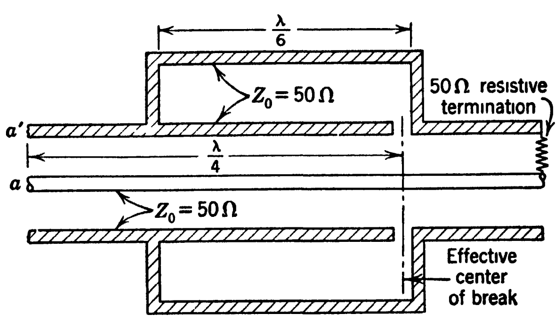


FIG 8·16 Section of a cylindrical loading section.

8·2 Determine the equivalent circuit of Figure 8·17.

8·3 Find the input impedance across aa' of Figure 8·17 when θ is equal to 90°.

8·4 Plot a curve of the magnitude of the impedance and its phase angle across aa' of Figure 8·17 as θ varies from 45 to 135 electrical degrees.

8·5 What value of Z_{OE} should be used in the conversion transformer shown in Figure 8·18?

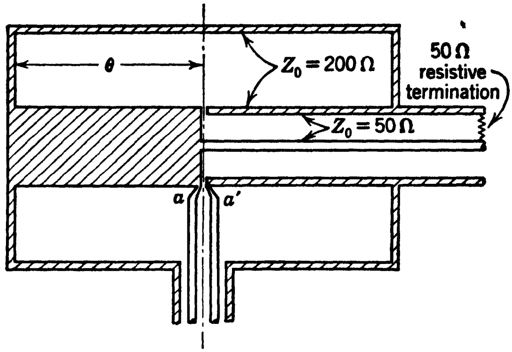


FIG. 8·17. A section of a symmetrical cylindrical coaxial to balanced line transformer.

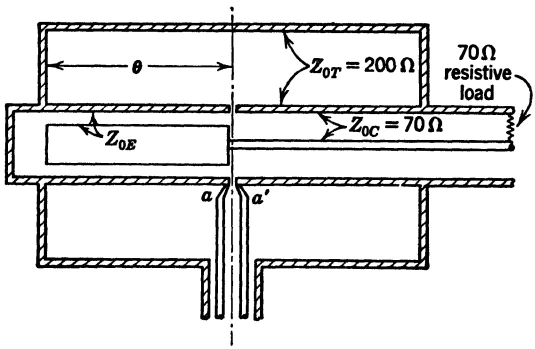


FIG. 8·18 A section of a compensated coaxial to balanced line transformer.

8·6 Using the result of problem 8·5, plot a curve of the magnitude and phase of the impedance across aa' of Figure 8·18 as θ varies from 45 to 135 electrical degrees.

CHARACTERISTIC IMPEDANCES OF TRANSMISSION LINES*

In these tabulations are listed the characteristic impedances of various configurations of transmission lines. In the formulas the following assumptions are made:

Formula assumptions

(1) The conductors are perfect, of unit permeability and imbedded in a vacuum (or, to a sufficient degree of accuracy, in air) so that the velocity of propagation is $3(10)^{10}$ cm per second.

Circular wire conductor radii

(2) The radii of circular wire conductors are small compared with the distance between them, or with the distance between them and the nearest extended surface (such as a plane, concentric cylinder, etc.)

Inductance-capacity formulas

From the characteristic impedance, the inductance and capacitance per unit length may be computed by means of the following formulas:

$$\text{Inductance: } L = 85Z_0 \text{ mmh per inch}$$

$$\text{Capacitance: } C = 85/Z_0 \text{ mmfd per inch}$$

where Z_0 is in ohms.

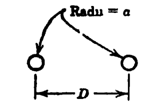
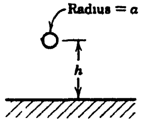
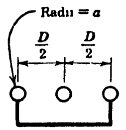
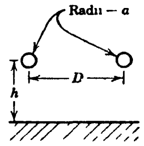
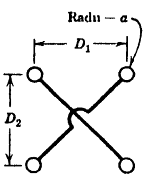
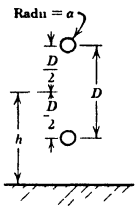
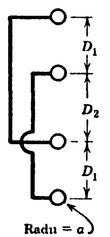
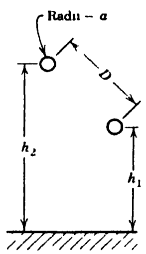
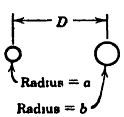
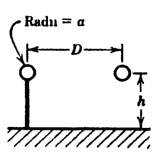
Dimensions are measured in any consistent units.

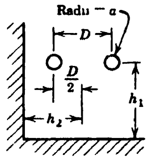
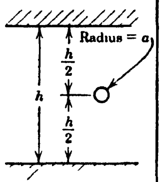
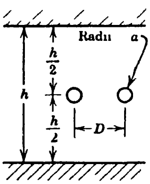
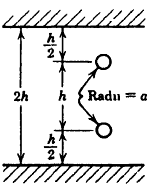
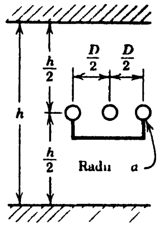
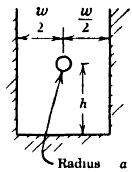
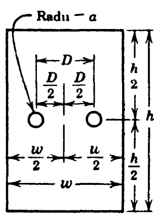
Reference data

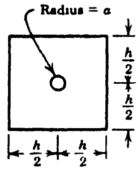
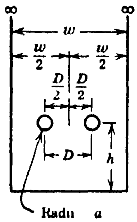
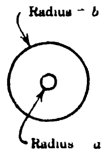
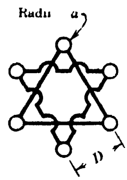
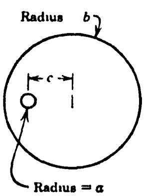
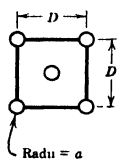
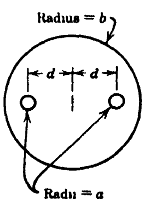
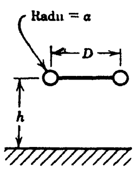
While some of the configurations listed may not appear to have any direct practical value, they are, nevertheless, useful as intermediate steps in computing the properties of practical configurations.†

* Sidney Frankel, "Characteristics Functions of Transmission Lines," *Communications*, March, 1943.

† See Sidney Frankel, "Characteristic Impedance of Parallel Wires in Rectangular Troughs.," *Proceedings of the I.R.E.*, Vol. 30, No. 4, April, 1942.

Configuration	Z_0 (Ohms)	Configuration	Z_0 (Ohms)
 Balanced 2 wire	$276 \log_{10} \frac{D}{a}$	 Single wire, ground return	$138 \log_{10} \frac{2h}{a}$
 3 wire	$207 \log_{10} \frac{D}{2.52a}$	 Balanced 2 wire near ground	$276 \log_{10} \left[\frac{2h}{a\sqrt{1+\left(\frac{2h}{D}\right)^2}}\right]$
 Balanced 4 wire	$138 \log_{10} \frac{D}{a\sqrt{1+\left(\frac{D}{D_1}\right)^2}}$	 Balanced 2 wire near ground	$276 \log_{10} \left[\frac{D\sqrt{1-\left(\frac{D}{2h}\right)}}{a}\right]$
 Two balanced 2-wire in parallel	$138 \log_{10} \frac{D_1}{a}\sqrt{1-\left(\frac{D_1}{D_1+D}\right)}$	 Balanced 2 wire near ground	$276 \log_{10} \left[\frac{D}{a} \cdot \frac{1}{\sqrt{1+\frac{D}{4h_1h}}}\right]$
 Balanced 2-wire, unequal radii	$276 \log_{10} \frac{D}{\sqrt{ab}}$	 2-wire, one wire grounded	$138 \frac{\left[\log_{10} \frac{D}{a}\right]\left[\log_{10} \frac{4h^2}{aD}\right]}{\log_{10} \frac{2h}{a}}$

Configuration	Z_0 (Ohms)
 Balanced 2 wire line near grounded corner	$276\log_{10}\left[\frac{D}{a}\sqrt{\frac{1-\left(\frac{D}{2h}\right)^2}{1+\left(\frac{D}{2h}\right)^2}}\right] - 69\log_{10}\left\{\left[1+\left(\frac{D}{m}\right)\right]-\left(\frac{D}{m}\right)\right\}$ where $m = h_1 + h_2$
 Single wire between grounded parallel planes ground return	$138\log_{10}\frac{2h}{\pi a}$
 Balanced line between grounded parallel planes	$276\log_{10}\left[\frac{2h\tanh\frac{\pi D}{2h}}{\pi a}\right]$
 Balanced line between grounded parallel planes	$276\log_{10}\frac{2h}{\pi a}$
 3 wire between parallel planes	$207\log_{10}\left[\frac{2h\tanh\frac{\pi D}{2h}}{\pi a\left(1+\operatorname{sech}^2\frac{\pi D}{2h}\right)^{1/3}}\right]$
 Single wire ground return	$138\log_{10}\left[\frac{2w\tanh\frac{\pi h}{w}}{\pi a}\right]$
 Balanced 2 wire line in rectangular section enclosure	$276\left\{\log_{10}\left[\frac{2h\tanh\frac{\pi D}{2h}}{\pi a}\right] - \sum_{m=1}^{\infty}\log_{10}\left[\frac{1+u_m^2}{1-v_m^2}\right]\right\}$ $u_m = \frac{\sinh\frac{\pi D}{2h}}{\cosh\frac{m\pi w}{2h}}$ $v_m = \frac{\sinh\frac{\pi D}{2h}}{\sinh\frac{m\pi w}{2h}}$

Configuration	Z_0 (Ohms)	Configuration	Z_0 (Ohms)
 "Concentric" line, square outer conductor	$138 \log_{10} \frac{0.539h}{a}$	 Balanced 2 wire line in semi infinite enclosure	$276 \log_{10} \left[\frac{w}{\pi a \sqrt{\operatorname{cosec}^2 \left(\frac{\pi D}{w} \right) + \operatorname{cosech}^2 \left(\frac{2\pi h}{w} \right)}} \right]$ *Note* — $\operatorname{cosec} x = \frac{1}{\sin x}$ $\operatorname{cosech} x = \frac{1}{\sinh x}$
 "Concentric" line	$138 \log_{10} \frac{b}{a}$ (Accurate for any a/b)	 Regular hexagon balanced to ground	$92 \log_{10} \frac{2D}{3a}$
 "Eccentric" line	$138 \log_{10} \frac{b^2 - c^2}{ab}$	 5-wire line	$172 \log_{10} \frac{D}{1.75a}$
 Balanced 2-wire "concentric" line	$276 \log_{10} \left[\frac{2d}{a} \left(\frac{1 - c^2}{1 + c^2} \right) \right]$ $c = \frac{d}{b}$	 2 wire parallel line, ground return	$69 \log_{10} \left[\frac{2h}{a} \sqrt{1 + \left(\frac{2h}{D} \right)^2} \right]$

Degs.	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157
1	0.0175	0.0192	0.0209	0.0227	0.0244	0.0262	0.0279	0.0297	0.0314	0.0332
2	0.0349	0.0367	0.0384	0.0401	0.0419	0.0436	0.0454	0.0471	0.0489	0.0506
3	0.0524	0.0541	0.0559	0.0576	0.0593	0.0611	0.0628	0.0646	0.0663	0.0681
4	0.0698	0.0716	0.0733	0.0750	0.0768	0.0785	0.0803	0.0820	0.0838	0.0855
5	0.0873	0.0890	0.0908	0.0925	0.0942	0.0960	0.0977	0.0995	0.1012	0.1030
6	0.1047	0.1065	0.1082	0.1100	0.1117	0.1134	0.1152	0.1169	0.1187	0.1204
7	0.1222	0.1239	0.1257	0.1274	0.1292	0.1309	0.1326	0.1344	0.1361	0.1379
8	0.1396	0.1414	0.1431	0.1449	0.1466	0.1484	0.1501	0.1518	0.1536	0.1553
9	0.1571	0.1588	0.1606	0.1623	0.1641	0.1658	0.1676	0.1693	0.1710	0.1728
10	0.1745	0.1763	0.1780	0.1798	0.1815	0.1833	0.1850	0.1868	0.1885	0.1902
11	0.1920	0.1937	0.1955	0.1972	0.1990	0.2007	0.2025	0.2042	0.2059	0.2077
12	0.2094	0.2112	0.2129	0.2147	0.2164	0.2182	0.2199	0.2217	0.2234	0.2251
13	0.2269	0.2286	0.2304	0.2321	0.2339	0.2356	0.2374	0.2391	0.2409	0.2426
14	0.2443	0.2461	0.2478	0.2496	0.2513	0.2531	0.2548	0.2566	0.2583	0.2601
15	0.2618	0.2635	0.2653	0.2670	0.2688	0.2705	0.2723	0.2740	0.2758	0.2775
16	0.2793	0.2810	0.2827	0.2845	0.2862	0.2880	0.2897	0.2915	0.2932	0.2950
17	0.2967	0.2985	0.3002	0.3019	0.3037	0.3054	0.3072	0.3089	0.3107	0.3124
18	0.3142	0.3159	0.3176	0.3194	0.3211	0.3229	0.3246	0.3264	0.3281	0.3299
19	0.3316	0.3334	0.3351	0.3368	0.3386	0.3403	0.3421	0.3438	0.3456	0.3473
20	0.3491	0.3508	0.3526	0.3543	0.3560	0.3578	0.3595	0.3613	0.3630	0.3648
21	0.3665	0.3683	0.3700	0.3718	0.3735	0.3752	0.3770	0.3787	0.3805	0.3822
22	0.3840	0.3857	0.3875	0.3892	0.3910	0.3927	0.3944	0.3962	0.3979	0.3997
23	0.4014	0.4032	0.4049	0.4067	0.4084	0.4102	0.4119	0.4136	0.4154	0.4171
24	0.4189	0.4206	0.4224	0.4241	0.4259	0.4276	0.4294	0.4311	0.4328	0.4346
25	0.4363	0.4381	0.4398	0.4416	0.4433	0.4451	0.4468	0.4485	0.4503	0.4520
26	0.4538	0.4555	0.4573	0.4590	0.4608	0.4625	0.4643	0.4660	0.4677	0.4695
27	0.4712	0.4730	0.4747	0.4765	0.4782	0.4800	0.4817	0.4835	0.4852	0.4869
28	0.4887	0.4904	0.4922	0.4939	0.4957	0.4974	0.4992	0.5009	0.5027	0.5044
29	0.5061	0.5079	0.5096	0.5114	0.5131	0.5149	0.5166	0.5184	0.5201	0.5219
30	0.5236	0.5253	0.5271	0.5288	0.5306	0.5323	0.5341	0.5358	0.5376	0.5393
31	0.5411	0.5428	0.5445	0.5463	0.5480	0.5498	0.5515	0.5533	0.5550	0.5568
32	0.5585	0.5603	0.5620	0.5637	0.5655	0.5672	0.5690	0.5707	0.5725	0.5742
33	0.5760	0.5777	0.5794	0.5812	0.5829	0.5847	0.5864	0.5882	0.5899	0.5917
34	0.5934	0.5952	0.5969	0.5986	0.6004	0.6021	0.6039	0.6056	0.6074	0.6091
35	0.6109	0.6126	0.6144	0.6161	0.6178	0.6196	0.6213	0.6231	0.6248	0.6266
36	0.6283	0.6301	0.6318	0.6336	0.6353	0.6370	0.6388	0.6405	0.6423	0.6440
37	0.6458	0.6475	0.6493	0.6510	0.6528	0.6545	0.6562	0.6580	0.6597	0.6615
38	0.6632	0.6650	0.6667	0.6685	0.6702	0.6720	0.6737	0.6754	0.6772	0.6789
39	0.6807	0.6824	0.6842	0.6859	0.6877	0.6894	0.6912	0.6929	0.6946	0.6964
40	0.6981	0.6999	0.7016	0.7034	0.7051	0.7069	0.7086	0.7103	0.7121	0.7138
41	0.7156	0.7173	0.7191	0.7208	0.7226	0.7243	0.7261	0.7278	0.7295	0.7313
42	0.7330	0.7348	0.7365	0.7383	0.7400	0.7418	0.7435	0.7453	0.7470	0.7487
43	0.7505	0.7522	0.7540	0.7557	0.7575	0.7592	0.7610	0.7627	0.7645	0.7662
44	0.7679	0.7697	0.7714	0.7732	0.7749	0.7767	0.7784	0.7802	0.7819	0.7837
45	0.7854	0.7871	0.7889	0.7906	0.7924	0.7941	0.7959	0.7976	0.7994	0.8011
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

$90° = 1.5708$ radians	$30° = \frac{\pi}{6}$, $45° = \frac{\pi}{4}$, $60° = \frac{\pi}{3}$, $90° = \frac{\pi}{2}$ radians
$180° = 3.1416$ radians	$120° = \frac{2\pi}{3}$, $135° = \frac{3\pi}{4}$, $150° = \frac{5\pi}{6}$, $180° = \pi$ radians
$270° = 4.7124$ radians	$210° = \frac{7\pi}{6}$, $225° = \frac{5\pi}{4}$, $240° = \frac{4\pi}{3}$, $270° = \frac{3\pi}{2}$ radians
$360° = 6.2832$ radians	$300° = \frac{5\pi}{3}$, $315° = \frac{7\pi}{4}$, $330° = \frac{11\pi}{6}$, $360° = 2\pi$ radians

* From Ralph G. Hudson, *The Engineers' Manual*, 2nd Ed., 1939, John Wiley and Sons, New York. Reprinted with permission.

Degs.	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
45	0.7854	0.7871	0.7889	0.7906	0.7924	0.7941	0.7959	0.7976	0.7994	0.8011
46	0.8029	0.8046	0.8063	0.8081	0.8098	0.8116	0.8133	0.8151	0.8168	0.8186
47	0.8203	0.8221	0.8238	0.8255	0.8273	0.8290	0.8308	0.8325	0.8343	0.8360
48	0.8378	0.8395	0.8412	0.8430	0.8447	0.8465	0.8482	0.8500	0.8517	0.8535
49	0.8552	0.8570	0.8587	0.8604	0.8622	0.8639	0.8657	0.8674	0.8692	0.8709
50	0.8727	0.8744	0.8762	0.8779	0.8796	0.8814	0.8831	0.8849	0.8866	0.8884
51	0.8901	0.8919	0.8936	0.8954	0.8971	0.8988	0.9006	0.9023	0.9041	0.9058
52	0.9076	0.9093	0.9111	0.9128	0.9146	0.9163	0.9180	0.9198	0.9215	0.9233
53	0.9250	0.9268	0.9285	0.9303	0.9320	0.9338	0.9355	0.9372	0.9390	0.9407
54	0.9425	0.9442	0.9460	0.9477	0.9495	0.9512	0.9529	0.9547	0.9564	0.9582
55	0.9599	0.9617	0.9634	0.9652	0.9669	0.9687	0.9704	0.9721	0.9739	0.9756
56	0.9774	0.9791	0.9809	0.9826	0.9844	0.9861	0.9879	0.9896	0.9913	0.9931
57	0.9948	0.9966	0.9983	1.0001	1.0018	1.0036	1.0053	1.0071	1.0088	1.0105
58	1.0123	1.0140	1.0158	1.0175	1.0193	1.0210	1.0228	1.0245	1.0263	1.0280
59	1.0297	1.0315	1.0332	1.0350	1.0367	1.0385	1.0402	1.0420	1.0437	1.0455
60	1.0472	1.0489	1.0507	1.0524	1.0542	1.0559	1.0577	1.0594	1.0612	1.0629
61	1.0647	1.0664	1.0681	1.0699	1.0716	1.0734	1.0751	1.0769	1.0786	1.0804
62	1.0821	1.0838	1.0856	1.0873	1.0891	1.0908	1.0926	1.0943	1.0961	1.0978
63	1.0996	1.1013	1.1030	1.1048	1.1065	1.1083	1.1100	1.1118	1.1135	1.1153
64	1.1170	1.1188	1.1205	1.1222	1.1240	1.1257	1.1275	1.1292	1.1310	1.1327
65	1.1345	1.1362	1.1380	1.1397	1.1414	1.1432	1.1449	1.1467	1.1484	1.1502
66	1.1519	1.1537	1.1554	1.1572	1.1589	1.1606	1.1624	1.1641	1.1659	1.1676
67	1.1694	1.1711	1.1729	1.1746	1.1764	1.1781	1.1798	1.1816	1.1833	1.1851
68	1.1868	1.1886	1.1903	1.1921	1.1938	1.1956	1.1973	1.1990	1.2008	1.2025
69	1.2043	1.2060	1.2078	1.2095	1.2113	1.2130	1.2147	1.2165	1.2182	1.2200
70	1.2217	1.2235	1.2252	1.2270	1.2287	1.2305	1.2322	1.2339	1.2357	1.2374
71	1.2392	1.2409	1.2427	1.2444	1.2462	1.2479	1.2497	1.2514	1.2531	1.2549
72	1.2566	1.2584	1.2601	1.2619	1.2636	1.2654	1.2671	1.2689	1.2706	1.2723
73	1.2741	1.2758	1.2776	1.2793	1.2811	1.2828	1.2846	1.2863	1.2881	1.2898
74	1.2915	1.2933	1.2950	1.2968	1.2985	1.3003	1.3020	1.3038	1.3055	1.3073
75	1.3090	1.3107	1.3125	1.3142	1.3160	1.3177	1.3195	1.3212	1.3230	1.3247
76	1.3265	1.3282	1.3299	1.3317	1.3334	1.3352	1.3369	1.3387	1.3404	1.3422
77	1.3439	1.3456	1.3474	1.3491	1.3509	1.3526	1.3544	1.3561	1.3579	1.3596
78	1.3614	1.3631	1.3648	1.3666	1.3683	1.3701	1.3718	1.3736	1.3753	1 3771
79	1.3788	1.3806	1.3823	1.3840	1.3858	1.3875	1.3893	1.3910	1.3928	1.3945
80	1.3963	1.3980	1.3998	1.4015	1.4032	1.4050	1.4067	1.4085	1.4102	1.4120
81	1.4137	1.4155	1.4172	1.4190	1.4207	1.4224	1.4242	1.4259	1.4277	1.4294
82	1.4312	1.4329	1.4347	1.4364	1.4382	1.4399	1.4416	1.4434	1.4451	1.4469
83	1.4486	1.4504	1.4521	1.4539	1.4556	1.4573	1.4591	1.4608	1.4626	1.4643
84	1.4661	1.4678	1.4696	1.4713	1.4731	1.4748	1.4765	1.4783	1.4800	1.4818
85	1.4835	1.4853	1.4870	1.4888	1.4905	1.4923	1.4940	1.4957	1.4975	1.4992
86	1.5010	1.5027	1.5045	1.5062	1.5080	1.5097	1.5115	1.5132	1.5149	1.5167
87	1.5184	1.5202	1.5219	1.5237	1.5254	1.5272	1.5289	1.5307	1.5324	1.5341
88	1.5359	1.5376	1.5394	1.5411	1.5429	1.5446	1.5464	1.5481	1.5499	1.5516
89	1.5533	1.5551	1.5568	1.5586	1.5603	1.5621	1.5638	1.5656	1.5673	1.5691
90	1.5708	1.5725	1.5743	1.5760	1.5778	1.5795	1.5813	1.5830	1.5848	1.5865
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

$90° = 1.5708$ radians	$30° = \frac{\pi}{6}$, $45° = \frac{\pi}{4}$, $60° = \frac{\pi}{3}$, $90° = \frac{\pi}{2}$ radians
$180° = 3.1416$ radians	$120° = \frac{2\pi}{3}$, $135° = \frac{3\pi}{4}$, $150° = \frac{5\pi}{6}$, $180° = \pi$ radians
$270° = 4.7124$ radians	$210° = \frac{7\pi}{6}$, $225° = \frac{5\pi}{4}$, $240° = \frac{4\pi}{3}$, $270° = \frac{3\pi}{2}$ radians
$360° = 6.2832$ radians	$300° = \frac{5\pi}{3}$, $315° = \frac{7\pi}{4}$, $330° = \frac{11\pi}{6}$, $360° = 2\pi$ radians

Degs.	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
0	sin	0 0000	0 0017	0 0035	0 0052	0 0070	0 0087	0 0105	0 0122	0 0140	0 0157
	cos	1 0000	1 0000	1 0000	1 0000	1 0000	1 0000	0 9999	0 9999	0 9999	0 9999
	tan	0 0000	0 0017	0 0035	0 0052	0 0070	0 0087	0 0105	0 0122	0 0140	0 0157
1	sin	0 0175	0 0192	0 0209	0 0227	0 0244	0 0262	0 0279	0 0297	0 0314	0 0332
	cos	0 9998	0 9998	0 9998	0 9997	0 9997	0 9997	0 9996	0 9996	0 9995	0 9995
	tan	0 0175	0 0192	0 0209	0 0227	0 0244	0 0262	0 0279	0 0297	0 0314	0 0332
2	sin	0 0349	0 0366	0 0384	0 0401	0 0419	0 0436	0 0454	0 0471	0 0488	0 0506
	cos	0 9994	0 9993	0 9993	0 9992	0 9991	0 9990	0 9990	0 9989	0 9988	0 9987
	tan	0 0349	0 0367	0 0384	0 0402	0 0419	0 0437	0 0454	0 0472	0 0489	0 0507
3	sin	0 0523	0 0541	0 0558	0 0576	0 0593	0 0610	0 0628	0 0645	0 0663	0 0680
	cos	0 9986	0 9985	0 9984	0 9983	0 9982	0 9981	0 9980	0 9979	0 9978	0 9977
	tan	0 0524	0 0542	0 0559	0 0577	0 0594	0 0612	0 0629	0 0647	0 0664	0 0682
4	sin	0 0698	0 0715	0 0732	0 0750	0 0767	0 0785	0 0802	0 0819	0 0837	0 0854
	cos	0 9976	0 9974	0 9973	0 9972	0 9971	0 9969	0 9968	0 9966	0 9965	0 9963
	tan	0 0699	0 0717	0 0734	0 0752	0 0769	0 0787	0 0805	0 0822	0 0840	0 0857
5	sin	0 0872	0 0889	0 0906	0 0924	0 0941	0 0958	0 0976	0 0993	0 1011	0 1028
	cos	0 9962	0 9960	0 9959	0 9957	0 9956	0 9954	0 9952	0 9951	0 9949	0 9947
	tan	0 0875	0 0892	0 0910	0 0928	0 0945	0 0963	0 0981	0 0998	0 1016	0 1033
6	sin	0 1045	0 1063	0 1080	0 1097	0 1115	0 1132	0 1149	0 1167	0 1184	0 1201
	cos	0 9945	0 9943	0 9942	0 9940	0 9938	0 9936	0 9934	0 9932	0 9930	0 9928
	tan	0 1051	0 1069	0 1086	0 1104	0 1122	0 1139	0 1157	0 1175	0 1192	0 1210
7	sin	0 1219	0 1236	0 1253	0 1271	0 1288	0 1305	0 1323	0 1340	0 1357	0 1374
	cos	0 9925	0 9923	0 9921	0 9919	0 9917	0 9914	0 9912	0 9910	0 9907	0 9905
	tan	0 1228	0 1246	0 1263	0 1281	0 1299	0 1317	0 1334	0 1352	0 1370	0 1388
8	sin	0 1392	0 1409	0 1426	0 1444	0 1461	0 1478	0 1495	0 1513	0 1530	0 1547
	cos	0 9903	0 9900	0 9898	0 9895	0 9893	0 9890	0 9888	0 9885	0 9882	0 9880
	tan	0 1405	0 1423	0 1441	0 1459	0 1477	0 1495	0 1512	0 1530	0 1548	0 1566
9	sin	0 1564	0 1582	0 1599	0 1616	0 1633	0 1650	0 1668	0 1685	0 1702	0 1719
	cos	0 9877	0 9874	0 9871	0 9869	0 9866	0 9863	0 9860	0 9857	0 9854	0 9851
	tan	0 1584	0 1602	0 1620	0 1638	0 1655	0 1673	0 1691	0 1709	0 1727	0 1745
10	sin	0 1736	0 1754	0 1771	0 1788	0 1805	0 1822	0 1840	0 1857	0 1874	0 1891
	cos	0 9848	0 9845	0 9842	0 9839	0 9836	0 9833	0 9829	0 9826	0 9823	0 9820
	tan	0 1763	0 1781	0 1799	0 1817	0 1835	0 1853	0 1871	0 1890	0 1908	0 1926
11	sin	0 1908	0 1925	0 1942	0 1959	0 1977	0 1994	0 2011	0 2028	0 2045	0 2062
	cos	0 9816	0 9813	0 9810	0 9806	0 9803	0 9799	0 9796	0 9792	0 9789	0 9785
	tan	0 1944	0 1962	0 1980	0 1998	0 2016	0 2035	0 2053	0 2071	0 2089	0 2107
12	sin	0 2079	0 2096	0 2113	0 2130	0 2147	0 2164	0 2181	0 2198	0 2215	0 2232
	cos	0 9781	0 9778	0 9774	0 9770	0 9767	0 9763	0 9759	0 9755	0 9751	0 9748
	tan	0 2126	0 2144	0 2162	0 2180	0 2199	0 2217	0 2235	0 2254	0 2272	0 2290
13	sin	0 2250	0 2267	0 2284	0 2300	0 2318	0 2334	0 2351	0 2368	0 2385	0 2402
	cos	0 9744	0 9740	0 9736	0 9732	0 9728	0 9724	0 9720	0 9715	0 9711	0 9707
	tan	0 2309	0 2327	0 2345	0 2364	0 2382	0 2401	0 2419	0 2438	0 2456	0 2475
14	sin	0 2419	0 2436	0 2453	0 2470	0 2487	0 2504	0 2521	0 2538	0 2554	0 2571
	cos	0 9703	0 9699	0 9694	0 9690	0 9686	0 9681	0 9677	0 9673	0 9668	0 9664
	tan	0 2493	0 2512	0 2530	0 2549	0 2568	0 2586	0 2605	0 2623	0 2642	0 2661
Degs.	Function	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

* From Ralph G Hudson, *The Engineers' Manual*, 2nd Ed., 1939, John Wiley and Sons, New York. Reprinted with permission.

Degs	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
15	sin	0 2588	0 2605	0 2622	0 2639	0 2656	0 2672	0 2689	0 2706	0 2723	0 2740
	cos	0 9659	0 9655	0 9650	0 9646	0 9641	0 9636	0 9632	0 9627	0 9622	0 9617
	tan	0 2679	0 2698	0.2717	0 2736	0 2754	0 2773	0 2792	0 2811	0 2830	0 2849
16	sin	0 2756	0 2773	0 2790	0 2807	0 2823	0 2840	0 2857	0 2874	0 2890	0 2907
	cos	0 9613	0 9608	0 9603	0 9598	0 9593	0 9588	0 9583	0 9578	0 9573	0 9568
	tan	0 2867	0 2886	0 2905	0 2924	0 2943	0 2962	0 2981	0 3000	0 3019	0 3038
17	sin	0 2924	0 2940	0 2957	0 2974	0 2990	0 3007	0 3024	0 3040	0 3057	0 3074
	cos	0 9563	0 9558	0 9553	0 9548	0 9542	0 9537	0 9532	0 9527	0 9521	0 9516
	tan	0 3057	0 3076	0 3096	0 3115	0 3134	0 3153	0 3172	0 3191	0 3211	0 3230
18	sin	0 3090	0 3107	0 3123	0 3140	0 3156	0 3173	0 3190	0 3206	0 3223	0 3239
	cos	0 9511	0 9505	0 9500	0 9494	0 9489	0 9483	0 9478	0 9472	0 9466	0 9461
	tan	0 3249	0 3269	0 3288	0 3307	0 3327	0 3346	0 3365	0 3385	0 3404	0 3424
19	sin	0 3256	0 3272	0 3289	0 3305	0 3322	0 3338	0 3355	0 3371	0 3387	0 3404
	cos	0 9455	0 9449	0 9444	0 9438	0 9432	0 9426	0 9421	0 9415	0 9409	0 9403
	tan	0 3443	0 3463	0 3482	0 3502	0 3522	0 3541	0 3561	0 3581	0 3600	0 3620
20	sin	0 3420	0 3437	0 3453	0 3469	0 3486	0 3502	0 3518	0 3535	0 3551	0 3567
	cos	0 9397	0 9391	0 9385	0 9379	0 9373	0 9367	0 9361	0 9354	0 9348	0 9342
	tan	0 3640	0 3659	0 3679	0 3699	0 3719	0 3739	0 3759	0 3779	0 3799	0 3819
21	sin	0 3584	0 3600	0 3616	0 3633	0 3649	0 3665	0 3681	0 3697	0 3714	0 3730
	cos	0 9336	0 9330	0 9323	0 9317	0 9311	0 9304	0 9298	0 9291	0 9285	0 9278
	tan	0 3839	0 3859	0 3879	0 3899	0 3919	0 3939	0 3959	0 3979	0 4000	0 4020
22	sin	0 3746	0 3762	0 3778	0 3795	0 3811	0 3827	0 3843	0 3859	0 3875	0 3891
	cos	0 9272	0 9265	0 9259	0 9252	0 9245	0 9239	0 9232	0 9225	0 9219	0 9212
	tan	0 4040	0 4061	0 4081	0 4101	0 4122	0 4142	0 4163	0 4183	0 4204	0 4224
23	sin	0 3907	0 3923	0 3939	0 3955	0 3971	0 3987	0 4003	0 4019	0 4035	0 4051
	cos	0 9205	0 9198	0 9191	0 9184	0 9178	0 9171	0 9164	0 9157	0 9150	0 9143
	tan	0 4245	0 4265	0 4286	0 4307	0 4327	0 4348	0 4369	0 4390	0 4411	0 4431
24	sin	0 4067	0 4083	0 4099	0 4115	0 4131	0 4147	0 4163	0 4179	0 4195	0 4210
	cos	0 9135	0 9128	0 9121	0 9114	0 9107	0 9100	0 9092	0 9085	0 9078	0 9070
	tan	0 4452	0 4473	0 4494	0 4515	0 4536	0 4557	0 4578	0 4599	0 4621	0 4642
25	sin	0 4226	0 4242	0 4258	0 4274	0 4289	0 4305	0 4321	0 4337	0 4352	0 4368
	cos	0 9063	0 9056	0 9048	0 9041	0 9033	0 9026	0 9018	0 9011	0 9003	0 8996
	tan	0 4663	0 4684	0 4706	0 4727	0 4748	0 4770	0 4791	0 4813	0 4834	0 4856
26	sin	0 4384	0 4399	0 4415	0 4431	0 4446	0 4462	0 4478	0 4493	0 4509	0 4524
	cos	0 8988	0 8980	0 8973	0 8965	0 8957	0 8949	0 8942	0 8934	0 8926	0 8918
	tan	0 4877	0 4899	0 4921	0 4942	0 4964	0 4986	0 5008	0 5029	0 5051	0 5073
27	sin	0 4540	0 4555	0 4571	0 4586	0 4602	0 4617	0 4633	0 4648	0 4664	0 4679
	cos	0 8910	0 8902	0 8894	0 8886	0 8878	0 8870	0 8862	0 8854	0 8846	0 8838
	tan	0 5095	0 5117	0 5139	0 5161	0 5184	0 5206	0 5228	0 5250	0 5272	0 5295
28	sin	0 4695	0 4710	0 4726	0 4741	0 4756	0 4772	0 4787	0 4802	0 4818	0 4833
	cos	0 8829	0 8821	0 8813	0 8805	0 8796	0 8788	0 8780	0 8771	0 8763	0 8755
	tan	0 5317	0 5340	0 5362	0 5384	0 5407	0 5430	0 5452	0 5475	0 5498	0 5520
29	sin	0 4848	0 4863	0 4879	0 4894	0 4909	0 4924	0 4939	0 4955	0 4970	0 4985
	cos	0 8746	0 8738	0 8729	0 8721	0 8712	0 8704	0 8695	0 8686	0 8678	0 8669
	tan	0 5543	0 5566	0 5589	0 5612	0 5635	0 5658	0 5681	0 5704	0 5727	0 5750
Degs.	Function	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

Degs.	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
30	sin	0 5000	0 5015	0 5030	0 5045	0 5060	0 5075	0 5090	0 5105	0 5120	0 5135
	cos	0 8660	0 8652	0 8643	0 8634	0 8625	0 8616	0 8607	0 8599	0 8590	0 8581
	tan	0 5774	0 5797	0 5820	0 5844	0 5867	0 5890	0 5914	0 5938	0 5961	0 5985
31	sin	0 5150	0 5165	0 5180	0 5195	0 5210	0 5225	0 5240	0 5255	0 5270	0 5284
	cos	0 857	0 8563	0 8554	0 8545	0 8536	0 8526	0 8517	0 8508	0 8499	0 8490
	tan	0 6009	0 6032	0 6056	0 6080	0 6104	0 6128	0 6152	0 6176	0 6200	0 6224
32	sin	0 5299	0 5314	0 5329	0 5344	0 5358	0 5373	0 5388	0 5402	0 5417	0 5432
	cos	0 8480	0 8471	0 8462	0 8453	0 8443	0 8434	0 8425	0 8415	0 8406	0 8396
	tan	0 6249	0 6273	0 6297	0 6322	0 6346	0 6371	0 6395	0 6420	0 6445	0 6469
33	sin	0 5446	0 5461	0 5476	0 5490	0 5505	0 5519	0 5534	0 5548	0 5563	0 5577
	cos	0 8387	0 8377	0 8368	0 8358	0 8348	0 8339	0 8329	0 8320	0 8310	0 8300
	tan	0 6494	0 6519	0 6544	0 6569	0 6594	0 6619	0 6644	0 6669	0 6694	0 6720
34	sin	0 5592	0 5606	0 5621	0 5635	0 5650	0 5664	0 5678	0 5693	0 5707	0 5721
	cos	0 8290	0 8281	0 8271	0 8261	0 8251	0 8241	0 8231	0 8221	0 8211	0 8202
	tan	0 6745	0 6771	0 6796	0 6822	0 6847	0 6873	0 6899	0 6924	0 6950	0 6976
35	sin	0 5736	0 5750	0 5764	0 5779	0 5793	0 5807	0 5821	0 5835	0 5850	0 5864
	cos	0 8192	0 8181	0 8171	0 8161	0 8151	0 8141	0 8131	0 8121	0 8111	0 8100
	tan	0 7002	0 7028	0 7054	0 7080	0 7107	0 7133	0 7159	0 7186	0 7212	0 7239
36	sin	0 5878	0 5892	0 5906	0 5920	0 5934	0 5948	0 5962	0 5976	0 5990	0 6004
	cos	0 8090	0 8080	0 8070	0 8059	0 8049	0 8039	0 8028	0 8018	0 8007	0 7997
	tan	0 7265	0 7292	0 7319	0 7346	0 7373	0 7400	0 7427	0 7454	0 7481	0 7508
37	sin	0 6018	0 603	0 6046	0 6060	0 6074	0 6088	0 6101	0 6115	0 6129	0 6143
	cos	0 7986	0 7976	0 7965	0 7955	0 7944	0 7934	0 7923	0 7912	0 7902	0 7891
	tan	0 7536	0 7563	0 7590	0 7618	0 7646	0 7673	0 7701	0 7729	0 7757	0 7785
38	sin	0 6157	0 6170	0 6184	0 6198	0 6211	0 6225	0 6239	0 6252	0 6266	0 6280
	cos	0 7880	0 786	0 7859	0 7848	0 7837	0 7826	0 7815	0 7804	0 7793	0 7782
	tan	0 7813	0 7841	0 7869	0 7898	0 7926	0 7954	0 7983	0 8012	0 8040	0 8069
39	sin	0 6293	0 6307	0 6320	0 6334	0 6347	0 6361	0 6374	0 6388	0 6401	0 6414
	cos	0 7771	0 7760	0 7749	0 7738	0 7727	0 7716	0 7705	0 7694	0 7683	0 7672
	tan	0 8098	0 8127	0 8156	0 8185	0 8214	0 8243	0 8273	0 8302	0 8332	0 8361
40	sin	0 6428	0 6441	0 6455	0 6468	0 6481	0 6494	0 6508	0 6521	0 6534	0 6547
	cos	0 7660	0 7649	0 7638	0 7627	0 7615	0 7604	0 7593	0 7581	0 7570	0 7559
	tan	0 8391	0 8421	0 8451	0 8481	0 8511	0 8541	0 8571	0 8601	0 8632	0 8662
41	sin	0 6561	0 6574	0 6587	0 6600	0 6613	0 6626	0 6639	0 6652	0 6665	0 6678
	cos	0 7547	0 7536	0 7524	0 7513	0 7501	0 7490	0 7478	0 7466	0 7455	0 7443
	tan	0 8693	0 8724	0 8754	0 8785	0 8816	0 8847	0 8878	0 8910	0 8941	0 8972
42	sin	0 6691	0 6704	0 6717	0 6730	0 6743	0 6756	0 6769	0 6782	0 6794	0 6807
	cos	0 7431	0 7420	0 7408	0 7396	0 7385	0 7373	0 7361	0 7349	0 7337	0 7325
	tan	0 9004	0 9036	0 9067	0 9099	0 9131	0 9163	0 9195	0 9228	0 9260	0 9293
43	sin	0 6820	0 6833	0 6845	0 6858	0 6871	0 6884	0 6896	0 6909	0 6921	0 6934
	cos	0 7314	0 7302	0 7290	0 7278	0 7266	0 7254	0 7242	0 7230	0 7218	0 7206
	tan	0 9325	0 9358	0 9391	0 9424	0 9457	0 9490	0 9523	0 9556	0 9590	0 9623
44	sin	0 6947	0 6959	0 6972	0 6984	0 6997	0 7009	0 7022	0 7034	0 7046	0 7059
	cos	0 7193	0 7181	0 7169	0 7157	0 7145	0 7133	0 7120	0 7108	0 7096	0 7083
	tan	0 9657	0 9691	0 9725	0 9759	0 9793	0 9827	0 9861	0 9896	0 9930	0 9965
Degs.	Function	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

Degs.	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
45	sin	0 7071	0 7083	0 7096	0 7108	0 7120	0 7133	0 7145	0 7157	0 7169	0 7181
	cos	0 7071	0 7059	0 7046	0 7034	0 7022	0 7009	0 6997	0 6984	0 6972	0 6959
	tan	1.0000	1 0035	1 0070	1 0105	1 0141	1 0176	1 0212	1 0247	1 0283	1 0319
46	sin	0 7193	0 7206	0 7218	0 7230	0 7242	0 7254	0 7266	0 7278	0 7290	0 7302
	cos	0 6947	0 6934	0 6921	0 6909	0 6896	0 6884	0 6871	0 6858	0 6845	0 6833
	tan	1 0355	1 0392	1 0428	1 0464	1 0501	1 0538	1 0575	1 0612	1 0649	1 0686
47	sin	0 7314	0 7325	0 7337	0 7349	0 7361	0 7373	0 7385	0 7396	0 7408	0 7420
	cos	0 6820	0 6807	0 6794	0 6782	0 6769	0 6756	0 6743	0 6730	0 6717	0 6704
	tan	1 0724	1 0761	1 0799	1 0837	1 0875	1 0913	1 0951	1 0990	1 1028	1 1067
48	sin	0 7431	0 7443	0 7455	0 7466	0 7478	0 7490	0 7501	0 7513	0 7524	0 7536
	cos	0 6691	0 6678	0 6665	0 6652	0 6639	0 6626	0 6613	0 6600	0 6587	0 6574
	tan	1 1106	1 1145	1 1184	1 1224	1 1263	1 1303	1 1343	1 1383	1 1423	1 1463
49	sin	0 7547	0 7559	0 7570	0 7581	0 7593	0 7604	0 7615	0 7627	0 7638	0 7649
	cos	0 6561	0 6547	0 6534	0 6521	0 6508	0 6494	0 6481	0 6468	0 6455	0 6441
	tan	1 1504	1 1544	1 1585	1 1626	1 1667	1 1708	1 1750	1 1792	1 1833	1 1875
50	sin	0 7660	0 7672	0 7683	0 7694	0 7705	0 7716	0 7727	0 7738	0 7749	0 7760
	cos	0 6428	0 6414	0 6401	0 6388	0 6374	0 6361	0 6347	0 6334	0 6320	0 6307
	tan	1 1918	1 1960	1 2002	1 2045	1 2088	1 2131	1 2174	1 2218	1 2261	1 2305
51	sin	0 7771	0 7782	0 7793	0 7804	0 7815	0 7826	0 7837	0 7848	0 7859	0 7869
	cos	0 6293	0 6280	0 6266	0 6252	0 6239	0 6225	0 6211	0 6198	0 6184	0 6170
	tan	1 2349	1 2393	1 2437	1 2482	1 2527	1 2572	1 2617	1 2662	1 2708	1 2753
52	sin	0 7880	0 7891	0 7902	0 7912	0 7923	0 7934	0 7944	0 7955	0 7965	0 7976
	cos	0 6157	0 6143	0 6129	0 6115	0 6101	0 6088	0 6074	0 6060	0 6046	0 6032
	tan	1 2799	1 2846	1 2892	1 2938	1 2985	1 3032	1 3079	1 3127	1 3175	1 3222
53	sin	0 7986	0 7997	0 8007	0 8018	0 8028	0 8039	0 8049	0 8059	0 8070	0 8080
	cos	0 6018	0 6004	0 5990	0 5976	0 5962	0 5948	0 5934	0 5920	0 5906	0 5892
	tan	1 3270	1 3319	1 3367	1 3416	1 3465	1 3514	1 3564	1 3613	1 3663	1 3713
54	sin	0 8090	0 8100	0 8111	0 8121	0 8131	0 8141	0 8151	0 8161	0 8171	0 8181
	cos	0 5878	0 5864	0 5850	0 5835	0 5821	0 5807	0 5793	0 5779	0 5764	0 5750
	tan	1 3764	1 3814	1 3865	1 3916	1 3968	1 4019	1 4071	1 4124	1 4176	1 4229
55	sin	0 8192	0 8202	0 8211	0 8221	0 8231	0 8241	0 8251	0 8261	0 8271	0 8281
	cos	0 5736	0 5721	0 5707	0 5693	0 5678	0 5664	0 5650	0 5635	0 5621	0 5606
	tan	1 4281	1 4335	1 4388	1 4442	1 4496	1 4550	1 4605	1 4659	1 4715	1 4770
56	sin	0 8290	0 8300	0 8310	0 8320	0 8329	0 8339	0 8348	0 8358	0 8368	0 8377
	cos	0 5592	0 5577	0 5563	0 5548	0 5534	0 5519	0 5505	0 5490	0 5476	0 5461
	tan	1 4826	1 4882	1 4938	1 4994	1 5051	1 5108	1 5166	1 5224	1 5282	1 5340
57	sin	0 8387	0 8396	0 8406	0 8415	0 8425	0 8434	0 8443	0 8453	0 8462	0 8471
	cos	0 5446	0 5432	0 5417	0 5402	0 5388	0 5373	0 5358	0 5344	0 5329	0 5314
	tan	1 5399	1 5458	1 5517	1 5577	1 5637	1 5697	1 5757	1 5818	1 5880	1 5941
58	sin	0 8480	0 8490	0 8499	0 8508	0 8517	0 8526	0 8536	0 8545	0 8554	0 8563
	cos	0 5299	0 5284	0 5270	0 5255	0 5240	0 5225	0 5210	0 5195	0 5180	0 5165
	tan	1 6003	1 6066	1 6128	1 6191	1 6255	1 6319	1 6383	1 6447	1 6512	1 6577
59	sin	0 8572	0 8581	0 8590	0 8599	0 8607	0 8616	0 8625	0 8634	0 8643	0 8652
	cos	0 5150	0 5135	0 5120	0 5105	0 5090	0 5075	0 5060	0 5045	0 5030	0 5015
	tan	1 6643	1 6709	1 6775	1 6842	1 6909	1 6977	1 7045	1 7113	1 7182	1 7251
Degs.	Function	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

Degs.	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
60	sin	0 8660	0 8669	0 8678	0 8686	0 8695	0 8704	0 8712	0 8721	0 8729	0 8738
	cos	0 5000	0 4985	0 4970	0 4955	0 4939	0 4924	0 4909	0 4894	0 4879	0 4863
	tan	1 7321	1 7391	1 7461	1 7532	1 7603	1 7675	1 7747	1 7820	1 7893	1 7966
61	sin	0 8746	0 8755	0 8763	0 8771	0 8780	0 8788	0 8796	0 8805	0 8813	0 8821
	cos	0 4848	0 4833	0 4818	0 4802	0 4787	0 4772	0 4756	0 4741	0 4726	0 4710
	tan	1 8040	1 8115	1 8190	1 8265	1 8341	1 8418	1 8495	1 8572	1 8650	1 8728
62	sin	0 8829	0 8838	0 8846	0 8854	0 8862	0 8870	0 8878	0 8886	0 8894	0 8902
	cos	0 4695	0 4679	0 4664	0 4648	0 4633	0 4617	0 4602	0 4586	0 4571	0 4555
	tan	1 8807	1 8887	1 8967	1 9047	1 9128	1 9210	1 9292	1 9375	1 9458	1 9542
63	sin	0 8910	0 8918	0 8926	0 8934	0 8942	0 8949	0 8957	0 8965	0 8973	0 8980
	cos	0 4540	0 4524	0 4509	0 4493	0 4478	0 4462	0 4446	0 4431	0 4415	0 4399
	tan	1 9626	1 9711	1 9797	1 9883	1 9970	2 0057	2 0145	2 0233	2 0323	2 0413
64	sin	0 8988	0 8996	0 9003	0 9011	0 9018	0 9026	0 9033	0 9041	0 9048	0 9056
	cos	0 4384	0 4368	0 4352	0 4337	0 4321	0 4305	0 4289	0 4274	0 4258	0 4242
	tan	2 0503	2 0594	2 0686	2 0778	2 0872	2 0965	2 1060	2 1155	2 1251	2 1348
65	sin	0 9063	0 9070	0 9078	0 9085	0 9092	0 9100	0 9107	0 9114	0 9121	0 9128
	cos	0 4226	0.4210	0 4195	0 4179	0 4163	0 4147	0 4131	0 4115	0 4099	0 4083
	tan	2 1445	2 1543	2 1642	2 1742	2 1842	2 1943	2 2045	2 2148	2 2251	2 2355
66	sin	0 9135	0 9143	0 9150	0 9157	0 9164	0 9171	0 9178	0 9184	0 9191	0 9198
	cos	0 4067	0 4051	0 4035	0 4019	0 4003	0 3987	0 3971	0 3955	0 3939	0 3923
	tan	2 2460	2 2566	2 2673	2 2781	2 2889	2 2998	2 3109	2 3220	2 3332	2 3445
67	sin	0 9205	0 9212	0 9219	0 9225	0 9232	0 9239	0 9245	0 9252	0 9259	0 9265
	cos	0 3907	0 3891	0 3875	0 3859	0 3843	0 3827	0 3811	0 3795	0 3778	0 3762
	tan	2 3559	2 3673	2 3789	2 3906	2 4023	2 4142	2 4262	2 4383	2 4504	2 4627
68	sin	0 9272	0 9278	0 9285	0 9291	0 9298	0 9304	0 9311	0 9317	0 9323	0 9330
	cos	0 3746	0 3730	0 3714	0 3697	0 3681	0 3665	0 3649	0 3633	0 3616	0 3600
	tan	2 4751	2 4876	2 5002	2 5129	2 5257	2 5386	2 5517	2 5649	2 5782	2 5916
69	sin	0 9336	0 9342	0 9348	0 9354	0 9361	0 9367	0 9373	0 9379	0 9385	0 9391
	cos	0 3584	0 3567	0 3551	0 3535	0 3518	0 3502	0 3486	0 3469	0 3453	0 3437
	tan	2 6051	2 6187	2 6325	2 6464	2 6605	2 6746	2 6889	2 7034	2 7179	2 7326
70	sin	0 9397	0 9403	0 9409	0 9415	0 9421	0 9426	0 9432	0 9438	0 9444	0 9449
	cos	0 3420	0 3404	0 3387	0 3371	0 3355	0 3338	0 3322	0 3305	0 3289	0 3272
	tan	2 7475	2 7625	2 7776	2 7929	2 8083	2 8239	2 8397	2 8556	2 8716	2 8878
71	sin	0 9455	0 9461	0 9466	0 9472	0 9478	0 9483	0 9489	0 9494	0 9500	0 9505
	cos	0 3256	0 3239	0 3223	0 3206	0 3190	0 3173	0 3156	0 3140	0 3123	0 3107
	tan	2 9042	2 9208	2 9375	2 9544	2 9714	2 9887	3 0061	3 0237	3 0415	3 0595
72	sin	0 9511	0 9516	0 9521	0 9527	0 9532	0 9537	0 9542	0 9548	0 9553	0 9558
	cos	0 3090	0 3074	0 3057	0 3040	0 3024	0 3007	0 2990	0 2974	0 2957	0 2940
	tan	3 0777	3 0961	3 1146	3 1334	3 1524	3 1716	3 1910	3 2106	3 2305	3 2506
73	sin	0 9563	0 9568	0 9573	0 9578	0 9583	0 9588	0 9593	0 9598	0 9603	0 9608
	cos	0 2924	0 2907	0 2890	0 2874	0 2857	0 2840	0 2823	0 2807	0 2790	0 2773
	tan	3 2709	3 2914	3 3122	3 3332	3 3544	3 3759	3 3977	3 4197	3 4420	3 4646
74	sin	0 9613	0 9617	0 9622	0 9627	0 9632	0 9636	0 9641	0 9646	0 9650	0 9655
	cos	0 2756	0 [illegible]740	0 2723	0 2706	0 2689	0 2672	0 2656	0 2639	0 2622	0 2605
	tan	3 4874	3 5105	3 5339	3 5576	3 5816	3 6059	3 6305	3 6554	3 6806	3 7062
Degs.	Function	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

Degs.	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
75	sin	0 9659	0 9664	0 9668	0 9673	0 9677	0 9681	0 9686	0 9690	0 9694	0 9699
	cos	0 2588	0 2571	0 2554	0 2538	0 2521	0 2504	0 2487	0 2470	0 2453	0 2436
	tan	3 7321	3 7583	3 7848	3 8118	3 8391	3 8667	3 8947	3 9232	3 9520	3 9812
76	sin	0 9703	0 9707	0 9711	0 9715	0 9720	0 9724	0 9728	0 9732	0 9736	0 9740
	cos	0 2419	0 2402	0 2385	0 2368	0 2351	0 2334	0 2317	0 2300	0 2284	0 2267
	tan	4 0108	4 0408	4 0713	4 1022	4 1335	4 1653	4 1976	4 2303	4 2635	4 2972
77	sin	0 9744	0 9748	0 9751	0 9755	0 9759	0 9763	0 9767	0 9770	0 9774	0 9778
	cos	0 2250	0 2232	0 2215	0 2198	0 2181	0 2164	0 2147	0 2130	0 2113	0 2096
	tan	4 3315	4 3662	4 4015	4 4374	4 4737	4 5107	4 5483	4 5864	4 6252	4 6646
78	sin	0 9781	0 9785	0 9789	0 9792	0 9796	0 9799	0 9803	0 9806	0 9810	0 9813
	cos	0 2079	0 2062	0 2045	0 2028	0 2011	0 1994	0 1977	0 1959	0 1942	0 1925
	tan	4 7046	4 7453	4 7867	4 8288	4 8716	4 9152	4 9594	5 0045	5 0504	5 0970
79	sin	0 9816	0 9820	0 9823	0 9826	0 9829	0 9833	0 9836	0 9839	0 9842	0 9845
	cos	0 1908	0 1891	0 1874	0 1857	0 1840	0 1822	0 1805	0 1788	0 1771	0 1754
	tan	5 1446	5 1929	5 2422	5 2924	5 3435	5 3955	5 4486	5 5026	5 5578	5 6140
80	sin	0 9848	0 9851	0 9854	0 9857	0 9860	0 9863	0 9866	0 9869	0 9871	0 9874
	cos	0 1736	0 1719	0 1702	0 1685	0 1668	0 1650	0 1633	0 1616	0 1599	0 1582
	tan	5 6713	5 7297	5 7894	5 8502	5 9124	5 9758	6 0405	6 1066	6 1742	6 2432
81	sin	0 9877	0 9880	0 9882	0 9885	0 9888	0 9890	0 9893	0 9895	0 9898	0 9900
	cos	0 1564	0 1547	0 1530	0 1513	0 1495	0 1478	0 1461	0 1444	0 1426	0 1409
	tan	6 3138	6 3859	6 4596	6 5350	6 6122	6 6912	6 7720	6 8548	6 9395	7 0264
82	sin	0 9903	0 9905	0 9907	0 9910	0 9912	0 9914	0 9917	0 9919	0 9921	0 9923
	cos	0 1392	0 1374	0 1357	0 1340	0 1323	0 1305	0 1288	0 1271	0 1253	0 1236
	tan	7 1154	7 2066	7 3002	7 3962	7 4947	7 5958	7 6996	7 8062	7 9158	8 0285
83	sin	0 9925	0 9928	0 9930	0 9932	0 9934	0 9936	0 9938	0 9940	0 9942	0 9943
	cos	0 1219	0 1201	0 1184	0 1167	0 1149	0 1132	0 1115	0 1097	0 1080	0 1063
	tan	8 1443	8 2636	8 3863	8 5126	8 6427	8 7769	8 9152	9 0579	9 2052	9 3572
84	sin	0 9945	0 9947	0 9949	0 9951	0 9952	0 9954	0 9956	0 9957	0 9959	0 9960
	cos	0 1045	0 1028	0 1011	0 0993	0 0976	0 0958	0 0941	0 0924	0 0906	0 0889
	tan	9 5144	9 6768	9 8448	10 02	10 20	10 39	10 58	10 [illegible]	10 99	11 20
85	sin	0 9962	0 9963	0 9965	0 9966	0 9968	0 9969	0 9971	0 9972	0 9973	0 9974
	cos	0 0872	0 0854	0 0837	0 0819	0 0802	0 0785	0 0767	0 0750	0 0732	0 0715
	tan	11 43	11 66	11 91	12 16	12 43	12 71	13 00	13 30	13 62	13 95
86	sin	0 9976	0 9977	0 9978	0 9979	0 9980	0 9981	0 9982	0 9983	0 9984	0 9985
	cos	0 0698	0 0680	0 0663	0 0645	0 0628	0 0610	0 0593	0 0576	0 0558	0 0541
	tan	14 30	14 67	15 06	15 46	15 89	16 35	16 83	17 34	17 89	18 46
87	sin	0 9986	0 9987	0 9988	0 9989	0 9990	0 9990	0 9991	0 9992	0 9993	0 9993
	cos	0 0523	0 0506	0 0488	0 0471	0 0454	0 0436	0 0419	0 0401	0 0384	0 0366
	tan	19 08	19 74	20 45	21 20	22 02	22 90	23 86	24 90	26 03	27 27
88	sin	0 9994	0 9995	0 9995	0 9996	0 9996	0 9997	0 9997	0 9997	0 9998	0 9998
	cos	0 0349	0 0332	0 0314	0 0297	0 0279	0 0262	0 0244	0 0227	0 0209	0 0192
	tan	28 64	30 14	31 82	33 69	35 80	38 19	40 92	44 07	47 74	52 08
89	sin	0 9998	0 9999	0 9999	0 9999	0 9999	1 000	1 000	1 000	1 000	1 000
	cos	0 0175	0 0157	0 0140	0 0122	0 0105	0 0087	0 0070	0 0052	0 0035	0 0017
	tan	57 29	63 66	71 62	81 85	95 49	114 6	143 2	191 0	286 5	573 0
Degs.	Function	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
0.00	**1.0000**	**1.0000**	**0.0000**	**1.0000**	**.00000**	**0.00000**	**$-\infty$**	**0.00000**	**$-\infty$**
0.01	1.0101	.99005	0.0100	1.0001	.01000	.00434	$\bar{2}$.00001	.00002	$\bar{3}$.99999
0.02	1.0202	.98020	0.0200	1.0002	.02000	.00869	.30106	.00009	$\bar{2}$.30097
0.03	1.0305	.97045	0.0300	1.0005	.02999	.01303	.47719	.00020	.47699
0.04	1.0408	.96079	0.0400	1.0008	.03998	.01737	.60218	.00035	.60183
0.05	1.0513	.95123	0.0500	1.0013	.04996	.02171	.69915	.00054	.69861
0.06	1.0618	.94176	0.0600	1.0018	.05993	.02606	.77841	.00078	.77763
0.07	1.0725	.93239	0.0701	1.0025	.06989	.03040	.84545	.00106	.84439
0.08	1.0833	.92312	0.0801	1.0032	.07983	.03474	.90355	.00139	.90216
0.09	1.0942	.91393	0.0901	1.0041	.08976	.03909	.95483	.00176	.95307
0.10	**1.1052**	**.90484**	**0.1002**	**1.0050**	**.09967**	**0.04343**	**$\bar{1}$.00072**	**0.00217**	**$\bar{2}$.99856**
0.11	1.1163	.89583	0.1102	1.0061	.10956	.04777	.04227	.00262	$\bar{1}$.03965
0.12	1.1275	.88692	0.1203	1.0072	.11943	.05212	.08022	.00312	.07710
0.13	1.1388	.87810	0.1304	1.0085	.12927	.05646	.11517	.00366	.11151
0.14	1.1503	.86936	0.1405	1.0098	.13909	.06080	.14755	.00424	.14330
0.15	1.1618	.86071	0.1506	1.0113	.14889	.06514	.17772	.00487	.17285
0.16	1.1735	.85214	0.1607	1.0128	.15865	.06949	.20597	.00554	.20044
0.17	1.1853	.84366	0.1708	1.0145	.16838	.07383	.23254	.00625	.22629
0.18	1.1972	.83527	0.1810	1.0162	.17808	.07817	.25762	.00700	.25062
0.19	1.2092	.82696	0.1911	1.0181	.18775	.08252	.28136	.00779	.27357
0.20	**1.2214**	**.81873**	**0.2013**	**1.0201**	**.19738**	**0.08686**	**$\bar{1}$.30392**	**0.00863**	**$\bar{1}$.29529**
0.21	1.2337	.81058	0.2115	1.0221	.20697	.09120	.32541	.00951	.31590
0.22	1.2461	.80252	0.2218	1.0243	.21652	.09554	.34592	.01043	.33549
0.23	1.2586	.79453	0.2320	1.0266	.22603	.09989	.36555	.01139	.35416
0.24	1.2712	.78663	0.2423	1.0289	.23550	.10423	.38437	.01239	.37198
0.25	1.2840	.77880	0.2526	1.0314	.24492	.10857	.40245	.01343	.38902
0.26	1.2969	.77105	0.2629	1.0340	.25430	.11292	.41986	.01452	.40534
0.27	1.3100	.76338	0.2733	1.0367	.26362	.11726	.43663	.01564	.42099
0.28	1.3231	.75578	0.2837	1.0395	.27291	.12160	.45282	.01681	.43601
0.29	1.3364	.74826	0.2941	1.0423	.28213	.12595	.46847	.01801	.45046
0.30	**1.3499**	**.74082**	**0.3045**	**1.0453**	**.29131**	**0.13029**	**1.48362**	**0.01926**	**$\bar{1}$.46436**
0.31	1.3634	.73345	0.3150	1.0484	.30044	.13463	.49830	.02054	.47775
0.32	1.3771	.72615	0.3255	1.0516	.30951	.13897	.51254	.02107	.49067
0.33	1.3910	.71892	0.3360	1.0549	.31852	.14332	.52637	.02323	.50314
0.34	1.4049	.71177	0.3466	1.0584	.32748	.14766	.53981	.02463	.51518
0.35	1.4191	.70469	0.3572	1.0619	.33638	.15200	.55290	.02607	.52682
0.36	1.4333	.69768	0.3678	1.0655	.34521	.15635	.56564	.02755	.53809
0.37	1.4477	.69073	0.3785	1.0692	.35399	.16069	.57807	.02907	.54899
0.38	1.4623	.68386	0.3892	1.0731	.36271	.16503	.59019	.03063	.55956
0.39	1.4770	.67706	0.4000	1.0770	.37136	.16937	.60202	03222	.56980
0.40	**1.4918**	**.67032**	**0.4108**	**1.0811**	**.37995**	**0.17372**	**$\bar{1}$.61358**	**0.03385**	**$\bar{1}$.57973**
0.41	1.5068	.66365	0.4216	1.0852	.38847	.17806	.62488	.03552	.58936
0.42	1.5220	.65705	0.4325	1.0895	.39693	.18240	.63594	.03723	.59871
0.43	1.5373.	.65051	0.4434	1.0939	.40532	.18675	.64677	.03897	.60780
0 44	1.5527	.64404	0.4543	1.0984	.41364	.19109	.65738	.04075	.61663
0.45	1.5683	.63763	0.4653	1.1030	.42190	.19543	.66777	.04256	.62521
0.46	1.5841	.63128	0.4764	1.1077	.43008	.19978	.67797	.04441	.63355
0.47	1.6000	.62500	0.4875	1.1125	.43820	.20412	.68797	.04630	.64167
0.48	1.6161	.61878	0.4986	1.1174	.44624	.20846	.69779	.04822	.64957
0.49	1.6323	.61263	0.5098	1.1225	.45422	.21280	.70744	.05018	.65726
0.50	**1.6487**	**.60653**	**0.5211**	**1.1276**	**.46212**	**0.21715**	**1.71692**	**0.05217**	**1.66475**
0.51	1.6653	.60050	0.5324	1.1329	.46995	.22149	.72624	.05419	.67205
0.52	1.6820	.59452	0.5438	1.1383	.47770	.22583	.73540	.05625	.67916
0.53	1.6989	.58860	0.5552	1.1438	.48538	.23018	.74442	.05834	.68608
0 54	1.7160	.58275	0.5666	1.1494	.49299	.23452	.75330	.06046	.69284
0.55	1.7333	.57695	0.5782	1.1551	.50052	.23886	.76204	.06262	.69942
0.56	1.7507	.57121	0.5897	1.1609	.50798	.24320	.77065	06481	.70584
0 57	1.7683	.56553	0.6014	1.1669	.51536	.24755	.77914	.06703	.71211
0 58	1.7860	.55990	0.6131	1.1730	.52267	.25189	.78751	.06929	.71822
0.59	1.8040	.55433	0.6248	1.1792	.52990	.25623	.79576	.07157	.72419
0.60	**1.8221**	**.54881**	**0.6367**	**1.1855**	**.53705**	**0.26058**	**$\bar{1}$.80390**	**0.07389**	**$\bar{1}$.73001**

* From Ovid W. Eshbach, *Handbook of Engineering Fundamentals*, 1936, John Wiley and Sons, New York. Reprinted with permission.

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
0.60	**1.8221**	**.54881**	**0.6367**	**1.1855**	**.53705**	**0.26058**	**$\bar{1}$.80390**	**0.07389**	**$\bar{1}$.73001**
0.61	1.8404	.54335	0.6485	1.1919	.54413	.26492	.81194	.07624	.73570
0.62	1.8589	.53794	0.6605	1.1984	.55113	.26926	.81987	.07861	.74125
0.63	1.8776	.53259	0.6725	1.2051	.55805	.27361	.82770	.08102	.74667
0.64	1.8965	.52729	0.6846	1.2119	.56490	.27795	.83543	.08346	.75197
0.65	1.9155	.52205	0.6967	1.2188	.57167	.28229	.84308	.08593	.75715
0.66	1.9348	.51685	0.7090	1.2258	.57836	.28663	.85063	.08843	.76220
0.67	1.9542	.51171	0.7213	1.2330	.58498	.29098	.85809	.09095	.76714
0.68	1.9739	.50662	0.7336	1.2402	.59152	.29532	.86548	.09351	.77197
0.69	1.9937	.50158	0.7461	1.2476	.59798	.29966	.87278	.09609	.77669
0.70	**2.0138**	**.49659**	**0.7586**	**1.2552**	**.60437**	**0.30401**	**$\bar{1}$.88000**	**0.09870**	**$\bar{1}$.78130**
0.71	2.0340	.49164	0.7712	1.2628	.61068	.30835	.88715	.10134	.78581
0.72	2.0544	.48675	0.7838	1.2706	.61691	.31269	.89423	.10401	.79022
0.73	2.0751	.48191	0.7966	1.2785	.62307	.31704	.90123	.10670	.79453
0.74	2.0959	.47711	0.8094	1.2865	.62915	.32138	.90817	.10942	.79875
0.75	2.1170	.47237	0.8223	1.2947	.63515	.32572	.91504	.11216	.80288
0.76	2.1383	.46767	0.8353	1.3030	.64108	.33006	.92185	.11493	.80691
0.77	2.1598	.46301	0.8484	1.3114	.64693	.33441	.92859	.11773	.81086
0.78	2.1815	.45841	0.8615	1.3199	.65271	.33875	.93527	.12055	.81472
0.79	2.2034	.45384	0.8748	1.3286	.65841	.34309	.94190	.12340	.81850
0.80	**2.2255**	**.44933**	**0.8881**	**1.3374**	**.66404**	**0.34744**	**$\bar{1}$.94846**	**0.12627**	**1.82219**
0.81	2.2479	.44486	0.9015	1.3464	.66959	.35178	.95498	.12917	.82581
0.82	2.2705	.44043	0.9150	1.3555	.67507	.35612	.96144	.13209	.82935
0.83	2.2933	.43605	0.9286	1.3647	.68048	.36046	.96784	.13503	.83281
0.84	2.3164	.43171	0.9423	1.3740	.68581	.36481	.97420	.13800	.83620
0.85	2.3396	.42741	0.9561	1.3835	.69107	.36915	.98051	.14099	.83952
0.86	2.3632	.42316	0.9700	1.3932	.69626	.37349	.98677	.14400	.84277
0.87	2.3869	.41895	0.9840	1.4029	.70137	.37784	.99299	.14704	.84595
0.88	2.4109	.41478	0.9981	1.4128	.70642	.38218	.99916	.15009	.84906
0.89	2.4351	.41066	1.0122	1.4229	.71139	.38652	0.00528	.15317	.85211
0.90	**2.4596**	**.40657**	**1.0265**	**1.4331**	**.71630**	**0.39087**	**0.01137**	**0.15627**	**$\bar{1}$.85509**
0.91	2.4843	.40252	1.0409	1.4434	.72113	.39521	.01741	.15939	.85801
0.92	2.5093	.39852	1.0554	1.4539	.72590	.39955	.02341	.16254	.86088
0.93	2.5345	.39455	1.0700	1.4645	.73059	.40389	.02937	.16570	.86368
0.94	2.5600	.39063	1.0847	1.4753	.73522	.40824	.03530	.16888	.86642
0.95	2.5857	.38674	1.0995	1.4862	.73978	.41258	.04119	.17208	.86910
0.96	2.6117	.38289	1.1144	1.4973	.74428	.41692	.04704	.17531	.87173
0.97	2.6379	.37908	1.1294	1.5085	.74870	.42127	.05286	.17855	.87431
0.98	2.6645	.37531	1.1446	1.5199	.75307	.42561	.05864	.18181	.87683
0.99	2.6912	.37158	1.1598	1.5314	.75736	.42995	.06439	.18509	.87930
1.00	**2.7183**	**.36788**	**1.1752**	**1.5431**	**.76159**	**0.43429**	**0.07011**	**0.18839**	**$\bar{1}$.88172**
1.01	2.7456	.36422	1.1907	1.5549	.76576	.43864	.07580	.19171	.88409
1.02	2.7732	.36059	1.2063	1.5669	.76987	.44298	.08146	.19504	.88642
1.03	2.8011	.35701	1.2220	1.5790	.77391	.44732	.08708	.19839	.88869
1.04	2.8292	.35345	1.2379	1.5913	.77789	.45167	.09268	.20176	.89092
1.05	2.8577	.34994	1.2539	1.6038	.78181	.45601	.09825	.20515	.89310
1.06	2.8864	.34646	1.2700	1.6164	.78566	.46035	.10379	.20855	.89524
1.07	2.9154	.34301	1.2862	1.6292	.78946	.46470	.10930	.21197	.89733
1.08	2.9447	.33960	1.3025	1.6421	.79320	.46904	.11479	.21541	.89938
1.09	2.9743	.33622	1.3190	1.6552	.79688	.47338	.12025	.21886	.90139
1.10	**3.0042**	**.33287**	**1.3356**	**1.6685**	**.80050**	**0.47772**	**0.12569**	**0.22233**	**$\bar{1}$.90336**
1.11	3.0344	.32956	1.3524	1.6820	.80406	.48207	.13111	.22582	.90529
1.12	3.0649	.32628	1.3693	1.6956	.80757	.48641	.13649	.22931	.90718
1.13	3.0957	.32303	1.3863	1.7093	.81102	.49075	.14186	.23283	.90903
1.14	3.1268	.31982	1.4035	1.7233	.81441	.49510	.14720	.23636	.91085
1.15	3.1582	.31664	1.4208	1.7374	.81775	.49944	.15253	.23990	.91262
1.16	3.1899	.31349	1.4382	1.7517	.82104	.50378	.15783	.24346	.91436
1.17	3.2220	.31037	1.4558	1.7662	.82427	.50812	.16311	.24703	.91607
1.18	3.2544	.30728	1.4735	1.7808	.82745	.51247	.16836	.25062	.91774
1.19	3.2871	.30422	1.4914	1.7957	.83058	.51681	.17360	.25422	.91938
1.20	**3.3201**	**.30119**	**1.5095**	**1.8107**	**.83365**	**0.52115**	**0.17882**	**0.25784**	**$\bar{1}$.92099**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
1.20	**3.3201**	**.30119**	**1.5095**	**1.8107**	**.83365**	**0.52115**	**0.17882**	**0.25784**	**1.92099**
1.21	3.3535	.29820	1.5276	1.8258	.83668	.52550	.18402	.26146	.92256
1.22	3.3872	.29523	1.5460	1.8412	.83965	.52984	.18920	.26510	.92410
1.23	3.4212	.29229	1.5645	1.8568	.84258	.53418	.19437	.26876	.92561
1.24	3.4556	.28938	1.5831	1.8725	.84546	.53853	.19951	.27242	.92709
1.25	3.4903	.28650	1.6019	1.8884	.84828	.54287	.20464	.27610	.92854
1.26	3.5254	.28365	1.6209	1.9045	.85106	.54721	.20975	.27979	.92996
1.27	3.5609	.28083	1.6400	1.9208	.85380	.55155	.21485	.28349	.93135
1.28	3.5966	.27804	1.6593	1.9373	.85648	.55590	.21993	.28721	.93272
1.29	3.6328	.27527	1.6788	1.9540	.85913	.56024	.22499	.29093	.93406
1.30	**3.6693**	**.27253**	**1.6984**	**1.9709**	**.86172**	**0.56458**	**0.23004**	**0.29467**	**$\bar{1}$.93537**
1.31	3.7062	.26982	1.7182	1.9880	.86428	.56893	.23507	.29842	.93665
1.32	3.7434	.26714	1.7381	2.0053	.86678	.57327	.24009	.30217	.93791
1.33	3.7810	.26448	1.7583	2.0228	.86925	.57761	.24509	.30594	.93914
1.34	3.8190	.26185	1.7786	2.0404	.87167	.58195	.25008	.30972	.94035
1.35	3.8574	.25924	1.7991	2.0583	.87405	.58630	.25505	.31352	.94154
1.36	3.8962	.25666	1.8198	2.0764	.87639	.59064	.26002	.31732	.94270
1.37	3.9354	.25411	1.8406	2.0947	.87869	.59498	.26496	.32113	.94384
1.38	3.9749	.25158	1.8617	2.1132	.88095	.59933	.26990	.32495	.94495
1.39	4.0149	.24908	1.8829	2.1320	.88317	.60367	.27482	.32878	.94604
1.40	**4.0552**	**.24660**	**1.9043**	**2.1509**	**.88535**	**0.60801**	**0.27974**	**0.33262**	**$\bar{1}$.94712**
1.41	4.0960	.24414	1.9259	2.1700	.88749	.61236	.28464	.33647	.94817
1.42	4.1371	.24171	1.9477	2.1894	.88960	.61670	.28952	.34033	.94919
1.43	4.1787	.23931	1.9697	2.2090	.89167	.62104	.29440	.34420	.95020
1.44	4.2207	.23693	1.9919	2.2288	.89370	.62538	.29926	.34807	.95119
1.45	4.2631	.23457	2.0143	2.2488	.89569	.62973	.30412	.35196	.95216
1.46	4.3060	.23224	2.0369	2.2691	.89765	.63407	.30896	.35585	.95311
1.47	4.3492	.22993	2.0597	2.2896	.89958	.63841	.31379	.35976	.95404
1.48	4.3929	.22764	2.0827	2.3103	.90147	.64276	.31862	.36367	.95495
1.49	4.4371	.22537	2.1059	2.3312	.90332	.64710	.32343	.36759	.95584
1.50	**4.4817**	**.22313**	**2.1293**	**2.3524**	**.90515**	**0.65144**	**0.32823**	**0.37151**	**$\bar{1}$.95672**
1.51	4.5267	.22091	2.1529	2.3738	.90694	.65578	.33303	.37545	.95758
1.52	4.5722	.21871	2.1768	2.3955	.90870	.66013	.33781	.37939	.95842
1.53	4.6182	.21654	2.2008	2.4174	.91042	.66447	.34258	.38334	.95924
1.54	4.6646	.21438	2.2251	2.4395	.91212	.66881	.34735	.38730	.96005
1.55	4.7115	.21225	2.2496	2.4619	.91379	.67316	.35211	.39126	.96084
1.56	4.7588	.21014	2.2743	2.4845	.91542	.67750	.35686	.39524	.96162
1.57	4.8066	.20805	2.2993	2.5073	.91703	.68184	.36160	.39921	.96238
1.58	4.8550	.20598	2.3245	2.5305	.91860	.68619	.36633	.40320	.96313
1.59	4.9037	.20393	2.3499	2.5538	.92015	.69053	.37105	.40719	.96386
1.60	**4.9530**	**.20190**	**2.3756**	**2.5775**	**.92167**	**0.69487**	**0.37577**	**0.41119**	**$\bar{1}$.96457**
1.61	5.0028	.19989	2.4015	2.6013	.92316	.69921	.38048	.41520	.96528
1.62	5.0531	.19790	2.4276	2.6255	.92462	.70356	.38518	.41921	.96597
1.63	5.1039	.19593	2.4540	2.6499	.92606	.70790	.38987	.42323	.96664
1.64	5.1552	.19398	2.4806	2.6746	.92747	.71224	.39456	.42725	.96730
1.65	5.2070	.19205	2.5075	2.6995	.92886	.71659	.39923	.43129	.96795
1.66	5.2593	.19014	2.5346	2.7247	.93022	.72093	.40391	.43532	.96858
1.67	5.3122	.18825	2.5620	2.7502	.93155	.72527	.40857	.43937	.96921
1.68	5.3656	.18637	2.5896	2.7760	.93286	.72961	.41323	.44341	.96982
1.69	5.4195	.18452	2.6175	2.8020	.93415	.73396	.41788	.44747	.97042
1.70	**5.4739**	**.18268**	**2.6456**	**2.8283**	**.93541**	**0.73830**	**0.42253**	**0.45153**	**$\bar{1}$.97100**
1.71	5.5290	.18087	2.6740	2.8549	.93665	.74264	.42717	.45559	.97158
1.72	5.5845	.17907	2.7027	2.8818	.93786	.74699	.43180	.45966	.97214
1.73	5.6407	.17728	2.7317	2.9090	.93906	.75133	.43643	.46374	.97269
1.74	5.6973	.17552	2.7609	2.9364	.94023	.75567	.44105	.46782	.97323
1.75	5.7546	.17377	2.7904	2.9642	.94138	.76002	.44567	.47191	.97376
1.76	5.8124	.17204	2.8202	2.9922	.94250	.76436	.45028	.47600	.97428
1.77	5.8709	.17033	2.8503	3.0206	.94361	.76870	.45488	.48009	.97479
1.78	5.9299	.16864	2.8806	3.0492	.94470	.77304	.45948	.48419	.97529
1.79	5.9895	.16696	2.9112	3.0782	.94576	.77739	.46408	.48830	.97578
1.80	**6.0496**	**.16530**	**2.9422**	**3.1075**	**.94681**	**0.78173**	**0.46867**	**0.49241**	**$\bar{1}$.97626**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
1.80	**6.0496**	**.16530**	**2.9422**	**3.1075**	**.94681**	**0.78173**	**0.46867**	**0.49241**	**$\bar{1}$.97626**
1.81	6.1104	.16365	2.9734	3.1371	.94783	.78607	.47325	.49652	.97673
1.82	6.1719	.16203	3.0049	3.1669	.94884	.79042	.47783	.50064	.97719
1.83	6.2339	.16041	3.0367	3.1972	.94983	.79476	.48241	.50476	.97764
1.84	6.2965	.15882	3.0689	3.2277	.95080	.79910	.48698	.50889	.97809
1.85	6.3598	.15724	3.1013	3.2585	.95175	.80344	.49154	.51302	.97852
1.86	6.4237	.15567	3.1340	3.2897	.95268	.80779	.49610	.51716	.97895
1.87	6.4883	.15412	3.1671	3.3212	.95359	.81213	.50066	.52130	.97936
1.88	6.5535	.15259	3.2005	3.3530	.95449	.81647	.50521	.52544	.97977
1.89	6.6194	.15107	3.2341	3.3852	.95537	.82082	.50976	.52959	.98017
1.90	**6.6859**	**.14957**	**3.2682**	**3.4177**	**.95624**	**0.82516**	**0.51430**	**0.53374**	**$\bar{1}$.98057**
1.91	6.7531	.14808	3.3025	3.4506	.95709	.82950	.51884	.53789	.98095
1.92	6.8210	.14661	3.3372	3.4838	.95792	.83385	.52338	.54205	.98133
1.93	6.8895	.14515	3.3722	3.5173	.95873	.83819	.52791	.54621	.98170
1.94	6.9588	.14370	3.4075	3.5512	.95953	.84253	.53244	.55038	.98206
1.95	7.0287	.14227	3.4432	3.5855	.96032	.84687	.53696	.55455	.98242
1.96	7.0993	.14086	3.4792	3.6201	.96109	.85122	.54148	.55872	.98272
1.97	7.1707	.13946	3.5156	3.6551	.96185	.85556	.54600	.56290	.98311
1.98	7.2427	.13807	3.5523	3.6904	.96259	.85990	.55051	.56707	.98344
1.99	7.3155	.13670	3.5894	3.7261	.96331	.86425	.55502	.57126	.98377
2.00	**7.3891**	**.13534**	**3.6269**	**3.7622**	**.96403**	**0.86859**	**0.55953**	**0.57544**	**$\bar{1}$.98409**
2.01	7.4633	.13399	3.6647	3.7987	.96473	.87293	.56403	.57963	.98440
2.02	7.5383	.13266	3.7028	3.8355	.96541	.87727	.56853	.58382	.98471
2.03	7.6141	.13134	3.7414	3.8727	.96609	.88162	.57303	.58802	.98502
2.04	7.6906	.13003	3.7803	3.9103	.96675	.88596	.57753	.59221	.98531
2.05	7.7679	.12873	3.8196	3.9483	.96740	.89030	.58202	.59641	.98560
2.06	7.8460	.12745	3.8593	3.9867	.96803	.89465	.58650	.60061	.98589
2.07	7.9248	.12619	3.8993	4.0255	.96865	.89899	.59099	.60482	.98617
2.08	8.0045	.12493	3.9398	4.0647	.96926	.90333	.59547	.60903	.98644
2.09	8.0849	.12369	3.9806	4.1043	.96986	.90768	.59995	.61324	.98671
2.10	**8.1662**	**.12246**	**4.0219**	**4.1443**	**.97045**	**0.91202**	**0.60443**	**0.61745**	**$\bar{1}$.98697**
2.11	8.2482	.12124	4.0635	4.1847	.97103	.91636	.60890	.62167	.98723
2.12	8.3311	.12003	4.1056	4.2256	.97159	.92070	.61337	.62589	.98748
2.13	8.4149	.11884	4.1480	4.2669	.97215	.92505	.61784	.63011	.98773
2.14	8.4994	.11765	4.1909	4.3085	.97269	.92939	.62231	.63433	.98798
2.15	8.5849	.11648	4.2342	4.3507	.97323	.93373	.62677	.63856	.98821
2.16	8.6711	.11533	4.2779	4.3932	.97375	.93808	.63123	.64278	.98845
2.17	8.7583	.11418	4.3221	4.4362	.97426	.94242	.63569	.64701	.98868
2.18	8.8463	.11304	4.3666	4.4797	.97477	.94676	.64015	.65125	.98890
2.19	8.9352	.11192	4.4116	4.5236	.97526	.95110	.64460	.65548	.98912
2.20	**9.0250**	**.11080**	**4.4571**	**4.5679**	**.97574**	**0.95545**	**0.64905**	**0.65972**	**$\bar{1}$.98934**
2.21	9.1157	.10970	4.5030	4.6127	.97622	.95979	.65350	.66396	.98955
2.22	9.2073	.10861	4.5494	4.6580	.97668	.96413	.65795	.66820	.98975
2.23	9.2999	.10753	4.5962	4.7037	.97714	.96848	.66240	.67244	.98996
2.24	9.3933	.10646	4.6434	4.7499	.97759	.97282	.66684	.67668	.99016
2.25	9.4877	.10540	4.6912	4.7966	.97803	.97716	.67128	.68093	.99035
2.26	9.5831	.10435	4.7394	4.8437	.97846	.98151	.67572	.68518	.99054
2.27	9.6794	.10331	4.7880	4.8914	.97888	.98585	.68016	.68943	.99073
2.28	9.7767	.10228	4.8372	4.9395	.97929	.99019	.68459	.69368	.99091
2.29	9.8749	.10127	4.8868	4.9881	.97970	.99453	.68903	.69794	.99109
2.30	**9.9742**	**.10026**	**4.9370**	**5.0372**	**.98010**	**0.99888**	**0.69346**	**0.70219**	**$\bar{1}$.99127**
2.31	10.074	.09926	4.9876	5.0868	.98049	1.00322	.69789	.70645	.99144
2.32	10.176	.09827	5.0387	5.1370	.98087	.00756	.70232	.71071	.99161
2.33	10.278	.09730	5.0903	5.1876	.98124	.01191	.70675	.71497	.99178
2.34	10.381	.09633	5.1425	5.2388	.98161	.01625	.71117	.71923	.99194
2.35	10.486	.09537	5.1951	5.2905	.98197	.02059	.71559	.72349	.99210
2.36	10.591	.09442	5.2483	5.3427	.98233	.02493	.72002	.72776	.99226
2.37	10.697	.09348	5.3020	5.3954	.98267	.02928	.72444	.73203	.99241
2.38	10.805	.09255	5.3562	5.4487	.98301	.03362	.72885	.73630	.99256
2.39	10.913	.09163	5.4109	5.5026	.98335	.03796	.73327	.74056	.99271
2.40	**11.023**	**.09072**	**5.4662**	**5.5569**	**.98367**	**1.04231**	**0.73769**	**0.74484**	**$\bar{1}$.9[illegible]85**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
2.40	**11.023**	**.09072**	**5.4662**	**5.5569**	**.98367**	**1.04231**	**0.73769**	**0.74484**	**$\bar{1}$.99285**
2.41	11.134	.08982	5.5221	5.6119	.98400	.04665	.74210	.74911	.99299
2.42	11.246	.08892	5.5785	5.6674	.98431	.05099	.74652	.75338	.99313
2.43	11.359	.08804	5.6354	5.7235	.98462	.05534	.75093	.75766	.99327
2.44	11.473	.08716	5.6929	5.7801	.98492	.05968	.75534	.76194	.99340
2.45	11.588	.08629	5.7510	5.8373	.98522	.06402	.75975	.76621	.99353
2.46	11.705	.08543	5.8097	5.8951	.98551	.06836	.76415	.77049	.99366
2.47	11.822	.08458	5.8689	5.9535	.98579	.07271	.76856	.77477	.99379
2.48	11.941	.08374	5.9288	6.0125	.98607	.07705	.77296	.77906	.99391
2.49	12.061	.08291	5.9892	6.0721	.98635	.08139	.77737	.78334	.99403
2.50	**12.182**	**.08208**	**6.0502**	**6.1323**	**.98661**	**1.08574**	**0.78177**	**0.78762**	**$\bar{1}$.99415**
2.51	12.305	.08127	6.1118	6.1931	.98688	.09008	.78617	.79191	.99426
2.52	12.429	.08046	6.1741	6.2545	.98714	.09442	.79057	.79619	.99438
2.53	12.554	.07966	6.2369	6.3166	.98739	.09877	.79497	.80048	.99449
2.54	12.680	.07887	6.3004	6.3793	.98764	.10311	.79937	.80477	.99460
2.55	12.807	.07808	6.3645	6.4426	.98788	.10745	.80377	.80906	.99470
2.56	12.936	.07730	6.4293	6.5066	.98812	.11179	.80816	.81335	.99481
2.57	13.066	.07654	6.4946	6.5712	.98835	.11614	.81256	.81764	.99491
2.58	13.197	.07577	6.5607	6.6365	.98858	.12048	.81695	.82194	.99501
2.59	13.330	.07502	6.6274	6.7024	.98881	.12482	.82134	.82623	.99511
2.60	**13.464**	**.07427**	**6.6947**	**6.7690**	**.98903**	**1.12917**	**0.82573**	**0.83052**	**$\bar{1}$.99521**
2.61	13.599	.07353	6.7628	6.8363	.98924	.13351	.83012	.83482	.99530
2.62	13.736	.07280	6.8315	6.9043	.98946	.13785	.83451	.83912	.99540
2.63	13.874	.07208	6.9008	6.9729	.98966	.14219	.83890	.84341	.99549
2.64	14.013	.07136	6.9709	7.0423	.98987	.14654	.84329	.84771	.99558
2.65	14.154	.07065	7.0417	7.1123	.99007	.15088	.84768	.85201	.99566
2.66	14.296	.06995	7.1132	7.1831	.99026	.15522	.85206	.85631	.99575
2.67	14.440	.06925	7.1854	7.2546	.99045	.15957	.85645	.86061	.99583
2.68	14.585	.06856	7.2583	7.3268	.99064	.16391	.86083	.86492	.99592
2.69	14.732	.06788	7.3319	7.3998	.99083	.16825	.86522	.86922	.99600
2.70	**14.880**	**.06721**	**7.4063**	**7.4735**	**.99101**	**1.17260**	**0.86960**	**0.87352**	**$\bar{1}$.99608**
2.71	15.029	.06654	7.4814	7.5479	.99118	.17694	.87398	.87783	.99615
2.72	15.180	.06587	7.5572	7.6231	.99136	.18128	.87836	.88213	.99623
2.73	15.333	.06522	7.6338	7.6991	.99153	.18562	.88274	.88644	.99631
2.74	15.487	.06457	7.7112	7.7758	.99170	.18997	.88712	.89074	.99638
2.75	15.643	.06393	7.7894	7.8533	.99186	.19431	.89150	.89505	.99645
2.76	15.800	.06329	7.8683	7.9316	.99202	.19865	.89588	.89936	.99652
2.77	15.959	.06266	7.9480	8.0106	.99218	.20300	.90026	.90367	.99659
2.78	16.119	.06204	8.0285	8.0905	.99233	.20734	.90463	.90798	.99666
2.79	16.281	.06142	8.1098	8.1712	.99248	.21168	.90901	.91229	.99672
2.80	**16.445**	**.06081**	**8.1919**	**8.2527**	**.99263**	**1.21602**	**0.91339**	**0.91660**	**$\bar{1}$.99679**
2.81	16.610	.06020	8.2749	8.3351	.99278	.22037	.91776	.92091	.99685
2.82	16.777	.05961	8.3586	8.4182	.99292	.22471	.92213	.92522	.99691
2.83	16.945	.05901	8.4432	8.5022	.99306	.22905	.92651	.92953	.99698
2.84	17.116	.05843	8.5287	8.5871	.99320	.23340	.93088	.93385	.99704
2.85	17.288	.05784	8.6150	8.6728	.99333	.23774	.93525	.93816	.99709
2.86	17.462	.05727	8.7021	8.7594	.99346	.24208	.93963	.94247	.99715
2.87	17.637	.05670	8.7902	8.8469	.99359	.24643	.94400	.94679	.99721
2.88	17.814	.05613	8.8791	8.9352	.99372	.25077	.94837	.95110	.99726
2.89	17.993	.05558	8.9689	9.0244	.99384	.25511	.95274	.95542	.99732
2.90	**18.174**	**.05502**	**9.0596**	**9.1146**	**.99396**	**1.25945**	**0.95711**	**0.95974**	**$\bar{1}$.99737**
2.91	18.357	.05448	9.1512	9.2056	.99408	.26380	.96148	.96405	.99742
2.92	18.541	.05393	9.2437	9.2976	.99420	.26814	.96584	.96837	.99747
2.93	18.728	.05340	9.3371	9.3905	.99431	.27248	.97021	.97269	.99752
2.94	18.916	.05287	9.4315	9.4844	.99443	.27683	.97458	.97701	.99757
2.95	19.106	.05234	9.5268	9.5791	.99454	.28117	.97895	.98133	.99762
2.96	19.298	.05182	9.6231	9.6749	.99464	.28551	.98331	.98565	.99767
2.97	19.492	.05130	9.7203	9.7716	.99475	.28985	.98768	.98997	.99771
2.98	19.688	.05079	9.8185	9.8693	.99485	.29420	.99205	.99429	.99776
2.99	19.886	.05029	9.9177	9.9680	.99496	.29854	.99641	.99861	.99780
3.	**20.086**	**.04979**	**10.018**	**10.068**	**.99505**	**1.30288**	**1.00078**	**1.00293**	**$\bar{1}$.99785**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
3.00	**20.086**	**.04979**	**10.018**	**10.068**	**.99505**	**1.30288**	**1.00078**	**1.00293**	**$\bar{1}$.99785**
3.01	20.287	.04929	10.119	10.168	.99515	.30723	.00514	.00725	.99789
3.02	20.491	.04880	10.221	10.270	.99525	.31157	.00950	.01157	.99793
3.03	20.697	.04832	10.325	10.373	.99534	.31591	.01387	.01589	.99797
3.04	20.905	.04783	10.429	10.477	.99543	.32026	.01823	.02022	.99801
3.05	21.115	.04736	10.534	10.581	.99552	.32460	.02259	.02454	.99805
3.06	21.328	.04689	10.640	10.687	.99561	.32894	.02696	.02886	.99809
3.07	21.542	.04642	10.748	10.794	.99570	.33328	.03132	.03319	.99813
3.08	21.758	.04596	10.856	10.902	.99578	.33763	.03568	.03751	.99817
3.09	21.977	.04550	10.966	11.011	.99587	.34197	.04004	.04184	.99820
3.10	**22.198**	**.04505**	**11.077**	**11.122**	**.99595**	**1.34631**	**1.04440**	**1.04616**	**$\bar{1}$.99824**
3.11	22.421	.04460	11.188	11.233	.99603	.35066	.04876	.05049	.99827
3.12	22.646	.04416	11.301	11.345	.99611	.35500	.05312	.05481	.99831
3.13	22.874	.04372	11.415	11.459	.99618	.35934	.05748	.05914	.99834
3.14	23.104	.04328	11.530	11.574	.99626	.36368	.06184	.06347	.99837
3.15	23.336	.04285	11.647	11.689	.99633	.36803	.06620	.06779	.99841
3.16	23.571	.04243	11.764	11.807	.99641	.37237	.07056	.07212	.99844
3.17	23.807	.04200	11.883	11.925	.99648	.37671	.07492	.07645	.99847
3.18	24.047	.04159	12.003	12.044	.99655	.38106	.07927	.08078	.99850
3.19	24.288	.04117	12.124	12.165	.99662	.38540	.08363	.08510	.99853
3.20	**24.533**	**.04076**	**12.246**	**12.287**	**.99668**	**1.38974**	**1.08799**	**1.08943**	**$\bar{1}$.99856**
3.21	24.779	.04036	12.369	12.410	.99675	.39409	.09235	.09376	.99859
3.22	25.028	.03996	12.494	12.534	.99681	.39843	.09670	.09809	.99861
3.23	25.280	.03956	12.620	12.660	.99688	.40277	.10106	.10242	.99864
3.24	25.534	.03916	12.747	12.786	.99694	.40711	.10542	.10675	.99867
3.25	25.790	.03877	12.876	12.915	.99700	.41146	.10977	.11108	.99869
3.26	26.050	.03839	13.006	13.044	.99706	.41580	.11413	.11541	.99872
3.27	26.311	.03801	13.137	13.175	.99712	.42014	.11849	.11974	.99875
3.28	26.576	.03763	13.269	13.307	.99717	.42449	.12284	.12407	.99877
3.29	26.843	.03725	13.403	13.440	.99723	.42883	.12720	.12840	.99879
3.30	**27.113**	**.03688**	**13.538**	**13.575**	**.99728**	**1.43317**	**1.13150**	**1.13273**	**$\bar{1}$.99882**
3.31	27.385	.03652	13.674	13.711	.99734	43751	.13591	.13706	.99884
3.32	27.660	.03615	13.812	13.848	.99739	.44186	.14026	.14139	.99886
3.33	27.938	.03579	13.951	13.987	.99744	.44620	.14461	.14573	.99889
3.34	28.219	.03544	14.092	14.127	.99749	.45054	.14897	.15006	.99891
3.35	28.503	.03508	14.234	14.269	.99754	.45489	.15332	.15439	.99893
3.36	28.789	.03474	14.377	14.412	.99759	.45923	.15768	.15872	.99895
3.37	29.079	.03439	14.522	14.556	.99764	.46357	.16203	.16306	.99897
3.38	29.371	.03405	14.668	14.702	.99768	.46792	.16638	.16739	.99899
3.39	29.666	.03371	14.816	14.850	.99773	.47226	.17073	.17172	.99901
3.40	**29.964**	**.03337**	**14.965**	**14.999**	**.99777**	**1.47660**	**1.17509**	**1.17605**	**$\bar{1}$.99903**
3.41	30.265	.03304	15.116	15.149	.99782	.48094	.17944	.18039	.99905
3.42	30.569	.03271	15.268	15.301	.99786	.48529	.18379	.18472	.99907
3.43	30.877	.03239	15.422	15.455	.99790	.48963	.18814	.18906	.99909
3.44	31.187	.03206	15.577	15.610	.99795	.49397	.19250	.19339	.99911
3.45	31.500	.03175	15.734	15.766	.99799	.49832	.19685	.19772	.99912
3.46	31.817	.03143	15.893	15.924	.99803	.50266	.20120	.20206	.99914
3.47	32.137	.03112	16.053	16.084	.99807	.50700	.20555	.20639	.99916
3.48	32.460	.03081	16.215	16.245	.99810	.51134	.20990	.21073	.99918
3.49	32.786	.03050	16.378	16.408	.99814	.51569	.21425	.21506	.99919
3.50	**33.115**	**.03020**	**16.543**	**16.573**	**.99818**	**1.52003**	**1.21860**	**1.21940**	**$\bar{1}$.99921**
3.51	33.448	.02990	16.709	16.739	.99821	.52437	.22296	.22373	.99922
3.52	33.784	.02960	16.877	16.907	.99825	.52872	.22731	.22807	.99924
3.53	34.124	.02930	17.047	17.077	.99828	.53306	.23166	.23240	.99925
3.54	34.467	.02901	17.219	17.248	.99832	.53740	.23601	.23674	.99927
3.55	34.813	.02872	17.392	17.421	.99835	.54175	.24036	.24107	.99928
3.56	35.163	.02844	17.567	17.596	.99838	.54609	.24471	.24541	.99930
3.57	35.517	.02816	17.744	17.772	.99842	.55043	.24906	.24975	.99931
3.58	35.874	.02788	17.923	17.951	.99845	.55477	.25341	.25408	.99933
3.59	36.234	.02760	18.103	18.131	.99848	.55912	.25776	.25842	.99934
3.60	**36.598**	**.02732**	**18.285**	**18.313**	**.99851**	**1.56346**	**1.26211**	**1.26275**	**$\bar{1}$.99935**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
3 60	**36 598**	**02732**	**18 285**	**18 313**	**99851**	**1 56346**	**1 26211**	**1 26275**	**$\bar{1}$ 99935**
3 61	36 966	02705	18 470	18 497	99854	56780	26646	26709	99936
3 62	37 338	02678	18 655	18 682	99857	57215	27080	27143	99938
3 63	37 713	02652	18 843	18 870	99859	57649	27515	27576	99939
3 64	38 092	02625	19 033	19 059	99862	58083	27950	28010	99940
3 65	38 475	02599	19 224	19 250	99865	58517	28385	28444	99941
3 66	38 861	02573	19 418	19 444	99868	58952	28820	28878	99942
3 67	39 252	02548	19 613	19 639	99870	59386	29255	29311	99944
3 68	39 646	02522	19 811	19 836	99873	59820	29690	29745	99945
3 69	40 045	02497	20 010	20 035	99875	60255	30125	30179	99946
3 70	**40 447**	**02472**	**20 211**	**20 236**	**99878**	**1 60689**	**1 30559**	**1 30612**	**$\bar{1}$ 99947**
3 71	40 854	02448	20 415	20 439	99880	61123	30994	31046	99948
3 72	41 264	02423	20 620	20 644	99883	61558	31429	31480	99949
3 73	41 679	02399	20 828	20 852	99885	61992	31864	31914	99950
3 74	42 098	02375	21 037	21 061	99887	62426	32299	32348	99951
3 75	42 521	02352	21 249	21 272	99889	62860	32733	32781	99952
3 76	42 948	02328	21 463	21 486	99892	63295	33168	33215	99953
3 77	43 380	02305	21 679	21 702	99894	63729	33603	33649	99954
3 78	43 816	02282	21 897	21 919	99896	64163	34038	34083	99955
3 79	44 256	02260	22 117	22 140	99898	64598	34472	34517	99956
3 80	**44 701**	**02237**	**22 339**	**22 362**	**99900**	**1 65032**	**1 34907**	**1 34951**	**$\bar{1}$ 99957**
3 81	45 150	02215	22 564	22 586	99902	65466	35342	35384	99957
3 82	45 604	02193	22 791	22 813	99904	65900	35777	35818	99958
3 83	46 063	02171	23 020	23 042	99906	66335	36211	36252	99959
3 84	46 525	02149	23 252	23 274	99908	66769	36646	36686	99960
3 85	46 993	02128	23 486	23 507	99909	67203	37081	37120	99961
3 86	47 465	02107	23 722	23 743	99911	67638	37515	37554	99961
3 87	47 942	02086	23 961	23 982	99913	68072	37950	37988	99962
3 88	48 424	02065	24 202	24 222	99915	68506	38385	38422	99963
3 89	48 911	02045	24 445	24 466	99916	68941	38819	38856	99964
3 90	**49 402**	**02024**	**24 691**	**24 711**	**99918**	**1 69375**	**1 39254**	**1 39290**	**$\bar{1}$ 99964**
3 91	49 899	02004	24 939	24 960	99920	69809	39689	39724	99965
3 92	50 400	01984	25 190	25 210	99921	70243	40123	40158	99966
3 93	50 907	01964	25 444	25 463	99923	70678	40558	40591	99966
3 94	51 419	01945	25 700	25 719	99924	71112	40993	41025	99967
3 95	51 935	01925	25 958	25 977	99926	71546	41427	41459	99968
3 96	52 457	01906	26 219	26 238	99927	71981	41862	41893	99968
3 97	52 985	01887	26 483	26 502	99929	72415	42296	42327	99969
3 98	53 517	01869	26 749	26 768	99930	72849	42731	42761	99970
3 99	54 055	01850	27 018	27 037	99932	73284	43166	43195	99970
4 00	**54 598**	**01832**	**27 290**	**27 308**	**99933**	**1 73718**	**1 43600**	**1 43629**	**$\bar{1}$ 99971**
4 01	55 147	01813	27 564	27 583	99934	74152	44035	44063	99971
4 02	55 701	01795	27 842	27 860	99936	74586	44469	44497	99972
4 03	56 261	01777	28 122	28 139	99937	75021	44904	44931	99973
4 04	56 826	01760	28 404	28 422	99938	75455	45339	45365	99973
4 05	57 397	01742	28 690	28 707	99939	75889	45773	45799	99974
4 06	57 974	01725	28 979	28 996	99941	76324	46208	46233	99974
4 07	58 557	01708	29 270	29 287	99942	76758	46642	46668	99975
4 08	59 145	01691	29 564	29 581	99943	77192	47077	47102	99975
4 09	59 740	01674	29 862	29 878	99944	77626	47511	47536	99976
4 10	**60 340**	**01657**	**30 162**	**30 178**	**99945**	**1 78061**	**1 47946**	**1 47970**	**$\bar{1}$ 99976**
4 11	60 947	01641	30 465	30 482	99946	78495	48380	48404	99977
4 12	61 559	01624	30 772	30 788	99947	78929	48815	48838	99977
4 13	62 178	01608	31 081	31 097	99948	79364	49249	49272	99978
4 14	62 803	01592	31 393	31 409	99949	79798	49684	49706	99978
4 15	63 434	01576	31 709	31 725	99950	80232	50118	50140	99978
4 16	64 072	01561	32 028	32 044	99951	80667	50553	50574	99979
4 17	64 715	01545	32 350	32 365	99952	81101	50987	51008	99979
4 18	65 366	01530	32 675	32 691	99953	81535	51422	51442	99980
4 19	66 023	01515	33 004	33 019	99954	81969	51856	51876	99980
4 20	**66 686**	**01500**	**33 336**	**33 351**	**99955**	**1 82404**	**1 52291**	**1 52310**	**$\bar{1}$ 99980**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
4.20	**66.686**	**.01500**	**33.336**	**33.351**	**.99955**	**1.82404**	**1.52291**	**1.52310**	**$\bar{1}$.99980**
4.21	67.357	.01485	33.671	33.686	.99956	.82838	.52725	.52745	.99981
4.22	68.033	.01470	34.009	34.024	.99957	.83272	.531 0	.53179	.99981
4.23	68.717	.01455	34.351	34.366	.99958	.83707	.53594	.53613	.99982
4.24	69.408	.01441	34.697	34.711	.99958	.84141	.54029	.54047	.99982
4.25	70.105	.01426	35.046	35.060	.99959	.84575	.54463	.54481	.99982
4.26	70.810	.01412	35.398	35.412	.99960	.85009	.54898	.54915	.99983
4.27	71.522	.01398	35.754	35.768	.99961	.85444	.55332	.55349	.99983
4.28	72.240	.01384	36.113	36.127	.99962	.85878	.55767	.55783	.99983
4.29	72.966	.01370	36.476	36.490	.99962	.86312	.56201	.56217	.99984
4.30	**73.700**	**.01357**	**36.843**	**36.857**	**.99963**	**1.86747**	**1.56636**	**1.56652**	**$\bar{1}$.99984**
4.31	74.440	.01343	37.214	37.227	.99964	.87181	.57070	.57086	.99984
4.32	75.189	.01330	37.588	37.601	.99965	.87615	.57505	.57520	.99985
4.33	75.944	.01317	37.966	37.979	.99965	.88050	.57939	.57954	.99985
4.34	76.708	.01304	38.347	38.360	.99966	.88484	.58373	.58388	.99985
4.35	77.478	.01291	38.733	38.746	.99967	.88918	.58808	.58822	.99986
4.36	78.257	.01278	39.122	39.135	.99967	.89352	.59242	.59256	.99986
4.37	79.044	.01265	39.515	39.528	.99968	.89787	.59677	.59691	.99986
4.38	79.838	.01253	39.913	39.925	.99969	.90221	.60111	.60125	.99986
4.39	80.640	.01240	40.314	40.326	.99969	.90655	.60546	.60559	.99987
4.40	**81.451**	**.01228**	**40.719**	**40.732**	**.99970**	**1.91090**	**1.60980**	**1.60993**	**$\bar{1}$.99987**
4.41	82.269	.01216	41.129	41.141	.99970	.91524	.61414	.61427	.99987
4.42	83.096	.01203	41.542	41.554	.99971	.91958	.61849	.61861	.99987
4.43	83.931	.01191	41.960	41.972	.99972	.92392	.62283	.62296	.99988
4.44	84.775	.01180	42.382	42.393	.99972	.92827	.62718	.62730	.99988
4.45	85.627	.01168	42.808	42.819	.99973	.93261	.63152	.63164	.99988
4.46	86.488	.01156	43.238	43.250	.99973	.93695	.63587	.63598	.99988
4.47	87.357	.01145	43.673	43.684	.99974	.94130	.64021	.64032	.99989
4.48	88.235	.01133	44.112	44.123	.99974	.94564	.64455	.64467	.99989
4.49	89.121	.01122	44.555	44.566	.99975	.94998	.64890	.64901	.99989
4.50	**90.017**	**.01111**	**45.003**	**45.014**	**.99975**	**1.95433**	**1.65324**	**1.65335**	**$\bar{1}$.99989**
4.51	90.922	.01100	45.455	45.466	.99976	.95867	.65759	.65769	.99989
4.52	91.836	.01089	45.912	45.923	.99976	.96301	.66193	.66203	.99990
4.53	92.759	.01078	46.374	46.385	.99977	.96735	.66627	.66637	.99990
4.54	93.691	.01067	46.840	46.851	.99977	.97170	67062	.67072	.99990
4.55	94.632	.01057	47.311	47.321	.99978	.97604	.67496	.67506	.99990
4.56	95.583	.01046	47.787	47.797	.99978	.98038	.67931	.67940	.99990
4.57	96.544	.01036	48.267	48 277	.99979	.98473	.68365	.68374	.99991
4.58	97.514	.01025	48.752	48.762	.99979	.98907	.68799	.68808	.99991
4.59	98.494	.01015	49.242	49.252	.99979	.99341	.69234	.69243	.99991
4.60	**99.484**	**.01005**	**49.737**	**49.747**	**.99980**	**1.99775**	**1.69668**	**1.69677**	**$\bar{1}$.99991**
4.61	100.48	.00995	50.237	50.247	.99980	2.00210	.70102	.70111	.99991
4.62	101.49	.00985	50.742	50.752	.99981	.00644	.70537	.70545	.99992
4.63	102.51	.00975	51.252	51.262	.99981	.01078	.70971	.70979	.99992
4.64	103.54	.00966	51.767	51.777	.99981	.01513	.71406	.71414	.99992
4.65	104.58	.00956	52.288	52.297	.99982	.01947	.71840	.71848	.99992
4.66	105.64	.00947	52.813	52.823	.99982	.02381	.72274	.72282	.99992
4.67	106.70	.00937	53.344	53.354	.99982	.02816	.72709	.72716	.99992
4.68	107.77	.00928	53.880	53.890	.99983	.03250	.73143	.73151	.99993
4.69	108.85	.00919	54.422	54.431	.99983	.03684	.73577	.73585	.99993
4.70	**109.95**	**.00910**	**54.969**	**54.978**	**.99983**	**2.04118**	**1.74012**	**1.74019**	**$\bar{1}$.99993**
4.71	111.05	.00900	55.522	55.531	.99984	.04553	.74446	.74453	.99993
4.72	112.17	.00892	56.080	56.089	.99984	.04987	.74881	.74887	.99993
4.73	113.30	.00883	56.643	56.652	.99984	.05421	.75315	.75322	.99993
4.74	114.43	.00874	57.213	57.222	.99985	.05856	.75749	.75756	.99993
4.75	115.58	.00865	57.788	57.796	.99985	.06290	.76184	.76190	.99993
4.76	116.75	.00857	58.369	58.377	.99985	.06724	.76618	.76624	.99994
4.77	117.92	.00848	58.955	58.964	.99986	.07158	.77052	.77059	.99994
4.78	119.10	.00840	59.548	59.556	.99986	.07593	.77487	.77493	.99994
4.79	120.30	.00831	60.147	60.155	.99986	.08027	.77921	.77927	.99994
4.80	**121.51**	**.00823**	**60.751**	**60.759**	**.99986**	**2.08461**	**1.78355**	**1.78361**	**$\bar{1}$.99994**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
4.80	**121.51**	**.00823**	**60.751**	**60.760**	**.99986**	**2.08461**	**1.78355**	**1.78361**	**$\bar{1}$.99994**
4.81	122.73	.00815	61.362	61.370	.99987	.08896	.78790	.78796	.99994
4.82	123.97	.00807	61.979	61.987	.99987	.09330	.79224	.79230	.99994
4.83	125.21	.00799	62.601	62.609	.99987	.09764	.79658	.79664	.99994
4.84	126.47	.00791	63.231	63.239	.99987	.10199	.80093	.80098	.99995
4.85	127.74	.00783	63.866	63.874	.99988	.10633	.80527	.80532	.99995
4.86	129.02	.00775	64.508	64.516	.99988	.11067	.80962	.80967	.99995
4.87	130.32	.00767	65.157	65.164	.99988	.11501	.81396	.81401	.99995
4.88	131.63	.00760	65.812	65.819	.99988	.11936	.81830	.81835	.99995
4.89	132.95	.00752	66.473	66.481	.99989	.12370	.82265	.82269	.99995
4.90	**134.29**	**.00745**	**67.141**	**67.149**	**.99989**	**2.12804**	**1.82699**	**1.82704**	**$\bar{1}$.99995**
4.91	135.64	.00737	67.816	67.823	.99989	.13239	.83133	.83138	.99995
4.92	137.00	.00730	68.498	68.505	.99989	.13673	.83568	.83572	.99995
4.93	138.38	.00723	69.186	69.193	.99990	.14107	.84002	.84006	.99995
4.94	139.77	.00715	69.882	69.889	.99990	.14541	.84436	.84441	.99996
4.95	141.17	.00708	70.584	70.591	.99990	.14976	.84871	.84875	.99996
4.96	142.59	.00701	71.293	71.300	.99990	.15410	.85305	.85309	.99996
4.97	144.03	.00694	72.010	72.017	.99990	.15844	.85739	.85743	.99996
4.98	145.47	.00687	72.734	72.741	.99991	.16279	.86174	.86178	.99996
4.99	146.94	.00681	73.465	73.472	.99991	.16713	.86608	.86612	.99996
5.00	**148.41**	**.00674**	**74.203**	**74.210**	**.99991**	**2.17147**	**1.87042**	**1.87046**	**$\bar{1}$.99996**
5.01	149.90	.00667	74.949	74.956	.99991	.17582	.87477	.87480	.99996
5.02	151.41	.00660	75.702	75.710	.99991	.18016	.87911	.87915	.99996
5.03	152.93	.00654	76.463	76.470	.99991	.18450	.88345	.88349	.99996
5.04	154.47	.00647	77.232	77.238	.99992	.18884	.88780	.88783	.99996
5.05	156.02	.00641	78.008	78.014	.99992	.19319	.89214	.89217	.99996
5.06	157.59	.00635	78.792	78.798	.99992	.19753	.89648	.89652	.99997
5.07	159.17	.00628	79.584	79.590	.99992	.20187	.90083	.90086	.99997
5.08	160.77	.00622	80.384	80.390	.99992	.20622	.90517	.90520	.99997
5.09	162.39	.00616	81.192	81.198	.99992	.21056	.90951	.90955	.99997
5.10	**164.02**	**.00610**	**82.008**	**82.014**	**.99993**	**2.21490**	**1.91386**	**1.91389**	**$\bar{1}$.99997**
5.11	165.67	.00604	82.832	82.838	.99993	.21924	.91820	.91823	.99997
5.12	167.34	.00598	83.665	83.671	.99993	.22359	.92254	.92257	.99997
5.13	169.02	.00592	84.506	84.512	.99993	.22793	.92689	.92692	.99997
5.14	170.72	.00586	85.355	85.361	.99993	.23227	.93123	.93126	.99997
5.15	172.43	.00580	86.213	86.219	.99993	.23662	.93557	.93560	.99997
5.16	174.16	.00574	87.079	87.085	.99993	.24096	.93992	.93994	.99997
5.17	175.91	.00568	87.955	87.960	.99994	.24530	.94426	.94429	.99997
5.18	177.68	.00563	88.839	88.844	.99994	.24965	.94860	.94863	.99997
5.19	179.47	.00557	89.732	89.737	.99994	.25399	.95294	.95297	.99997
5.20	**181.27**	**.00552**	**90.633**	**90.639**	**.99994**	**2.25833**	**1.95729**	**1.95731**	**$\bar{1}$.99997**
5.21	183.09	.00546	91.544	91.550	.99994	.26267	.96163	.96166	.99997
5.22	184.93	.00541	92.464	92.470	.99994	.26702	.96597	.96600	.99997
5.23	186.79	.00535	93.394	93.399	.99994	.27136	.97032	.97034	.99998
5.24	188.67	.00530	94.332	94.338	.99994	.27570	.97466	.97469	.99998
5.25	190.57	.00525	95.281	95.286	.99994	.28005	.97900	.97903	.99998
5.26	192.48	.00520	96.238	96.243	.99995	.28439	.98335	.98337	.99998
5.27	194.42	.00514	97.205	97.211	.99995	.28873	.98769	.98771	.99998
5.28	196.37	.00509	98.182	98.188	.99995	.29307	.99203	.99206	.99998
5.29	198.34	.00504	99.169	99.174	.99995	.29742	.99638	.99640	.99998
5.30	**200.34**	**.00499**	**100.17**	**100.17**	**.99995**	**2.30176**	**2.00072**	**2.00074**	**$\bar{1}$.99998**
5.31	202.35	.00494	101.17	101.18	.99995	.30610	.00506	.00508	.99998
5.32	204.38	.00489	102.19	102.19	.99995	.31045	.00941	.00943	.99998
5.33	206.44	.00484	103.22	103.22	.99995	.31479	.01375	.01377	.99998
5.34	208.51	.00480	104.25	104.26	.99995	.31913	.01809	.01811	.99998
5.35	210.61	.00475	105.30	105.31	.99995	.32348	.02244	.02246	.99998
5.36	212.72	.00470	106.36	106.36	.99996	.32782	.02678	.02680	.99998
5.37	214.86	.00465	107.43	107.43	.99996	.33216	.03112	.03114	.99998
5.38	217.02	.00461	108.51	108.51	.99996	.33650	.03547	.03548	.99998
5.39	219.20	.00456	109.60	109.60	.99996	.34085	.03981	.03983	.99998
5.40	**221.41**	**.00452**	**110.70**	**110.71**	**.99996**	**2.34519**	**2.04415**	**2.04417**	**$\bar{1}$.99998**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
5.40	**221.41**	**.00452**	**110.70**	**110.71**	**.99996**	**2.34519**	**2.04415**	**2.04417**	**$\bar{1}$.99998**
5.41	223.63	.00447	111.81	111.82	.99996	.34953	.04849	.04851	.99998
5.42	225.88	.00443	112.94	112.94	.99996	.35388	.05284	.05285	.99998
5.43	228.15	.00438	114.07	114.08	.99996	.35822	.05718	.05720	.99998
5.44	230.44	.00434	115.22	115.22	.99996	.36256	.06152	.06154	.99998
5.45	232.76	.00430	116.38	116.38	.99996	.36690	.06587	.06588	.99998
5.46	235.10	.00425	117.55	117.55	.99996	.37125	.07021	.07023	.99998
5.47	237.46	.00421	118.73	118.73	.99996	.37559	.07455	.07457	.99998
5.48	239.85	.00417	119.92	119.93	.99997	.37993	.07890	.07891	.99998
5.49	242.26	.00413	121.13	121.13	.99997	.38428	.08324	.08325	.99999
5.50	**244.69**	**.00409**	**122.34**	**122.35**	**.99997**	**2.38862**	**2.08758**	**2.08760**	**$\bar{1}$.99999**
5.51	247.15	.00405	123.57	123.58	.99997	.39296	.09193	.09194	.99999
5.52	249.64	.00401	124.82	124.82	.99997	.39731	.09627	.09628	.99999
5.53	252.14	.00397	126.07	126.07	.99997	.40165	.10061	.10063	.99999
5.54	254.68	.00393	127.34	127.34	.99997	.40599	.10495	.10497	.99999
5.55	257.24	.00389	128.62	128.62	.99997	.41033	.10930	.10931	.99999
5.56	259.82	.00385	129.91	129.91	.99997	.41468	.11364	.11365	.99999
5.57	262.43	.00381	131.22	131.22	.99997	.41902	.11798	.11800	.99999
5.58	265.07	.00377	132.53	132.54	.99997	.42336	.12233	.12234	.99999
5.59	267.74	.00374	133.87	133.87	.99997	.42771	.12667	.12668	.99999
5.60	**270.43**	**.00370**	**135.21**	**135.22**	**.99997**	**2.43205**	**2.13101**	**2.13103**	**$\bar{1}$.99999**
5.61	273.14	.00366	136.57	136.57	.99997	.43639	.13536	.13537	.99999
5.62	275.89	.00362	137.94	137.95	.99997	.44074	.13970	.13971	.99999
5.63	278.66	.00359	139.33	139.33	.99997	.44508	.14404	.14405	.99999
5.64	281.46	.00355	140.73	140.73	.99997	.44942	.14839	.14840	.99999
5.65	284.29	.00352	142.14	142.15	.99998	.45376	.15273	.15274	.99999
5.66	287.15	.00348	143.57	143.58	.99998	.45811	.15707	.15708	.99999
5.67	290.03	.00345	145.02	145.02	.99998	.46245	.16141	.16142	.99999
5.68	292.95	.00341	146.47	146.48	.99998	.46679	.16576	.16577	.99999
5.69	295.89	.00338	147.95	147.95	.99998	.47114	.17010	.17011	.99999
5.70	**298.87**	**.00335**	**149.43**	**149.44**	**.99998**	**2.47548**	**2.17444**	**2.17445**	**$\bar{1}$.99999**
5.71	301.87	.00331	150.93	150.94	.99998	.47982	.17879	.17880	.99999
5.72	304.90	.00328	152.45	152.45	.99998	.48416	.18313	.18314	.99999
5.73	307.97	.00325	153.98	153.99	.99998	.48851	.18747	.18748	.99999
5.74	311.06	.00321	155.53	155.53	.99998	.49285	.19182	.19182	.99999
5.75	314.19	.00318	157.09	157.10	.99998	.49719	.19616	.19617	.99999
5.76	317.35	.00315	158.67	158.68	.99998	.50154	.20050	.20051	.99999
5.77	320.54	.00312	160.27	160.27	.99998	.50588	.20484	.20485	.99999
5.78	323.76	.00309	161.88	161.88	.99998	.51022	.20919	.20920	.99999
5.79	327.01	.00306	163.51	163.51	.99998	.51457	.21353	.21354	.99999
5.80	**330.30**	**.00303**	**165.15**	**165.15**	**.99998**	**2.51891**	**2.21787**	**2.21788**	**$\bar{1}$.99999**
5.81	333.62	.00300	166.81	166.81	.99998	.52325	.22222	.22222	.99999
5.82	336.97	.00297	168.48	168.49	.99998	.52759	.22656	.22657	.99999
5.83	340.36	.00294	170.18	170.18	.99998	.53194	.23090	.23091	.99999
5.84	343.78	.00291	171.89	171.89	.99998	.53628	.23525	.23525	.99999
5.85	347.23	.00288	173.62	173.62	.99998	.54062	.23959	.23960	.99999
5.86	350.72	.00285	175.36	175.36	.99998	.54497	.24393	.24394	.99999
5.87	354.25	.00282	177.12	177.13	.99998	.54931	.24828	.24828	.99999
5.88	357.81	.00279	178.90	178.91	.99998	.55365	.25262	.25262	.99999
5.89	361.41	.00277	180.70	180.70	.99998	.55799	.25696	.25697	.99999
5.90	**365.04**	**.00274**	**182.52**	**182.52**	**.99998**	**2.56234**	**2.26130**	**2.26131**	**$\bar{1}$.99999**
5.91	368.71	.00271	184.35	184.35	.99999	.56668	.26565	.26565	.99999
5.92	372.41	.00269	186.20	186.21	.99999	.57102	.26999	.27000	.99999
5.93	376.15	.00266	188.08	188.08	.99999	.57537	.27433	.27434	.99999
5.94	379.93	.00263	189.97	189.97	.99999	.57971	.27868	.27868	.99999
5.95	383.75	.00261	191.88	191.88	.99999	.58405	.28302	.28303	.99999
5.96	387.61	.00258	193.80	193.81	.99999	.58840	.28736	.28737	.99999
5.97	391.51	.00255	195.75	195.75	.99999	.59274	.29171	.29171	.99999
5.98	395.44	.00253	197.72	197.72	.99999	.59708	.29605	.29605	.99999
5.99	399.41	.00250	199.71	199.71	.99999	.60142	.30039	.30040	.99999
6.00	**403.43**	**.00248**	**201.71**	**201.72**	**.99999**	**2.60577**	**2.30473**	**2.30474**	**$\bar{1}$.99999**

BESSEL FUNCTIONS*

TABLE 1

$J_0(z)$

z	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	1.0000	0.9975	0.9900	0.9776	0.9604	0.9385	0.9120	0.8812	0.8463	0.8075
1	0.7652	0.7196	0.6711	0.6201	0.5669	0.5118	0.4554	0.3980	0.3400	0.2818
2	0.2239	0.1666	0.1104	0.0555	0.0025	−0.0484	−0.0968	−0.1424	−0.1850	−0.2243
3	−0.2601	−0.2921	−0.3202	−0.3443	−0.3643	−0.3801	−0.3918	−0.3992	−0.4026	−0.4018
4	−0.3971	−0.3887	−0.3766	−0.3610	−0.3423	−0.3205	−0.2961	−0.2693	−0.2404	−0.2097
5	−0.1776	−0.1443	−0.1103	−0.0758	−0.0412	−0.0068	+0.0270	0.0599	0.0917	0.1220
6	0.1506	0.1773	0.2017	0.2238	0.2433	0.2601	0.2740	0.2851	0.2931	0.2981
7	0.3001	0.2991	0.2951	0.2882	0.2786	0.2663	0.2516	0.2346	0.2154	0.1944
8	0.1717	0.1475	0.1222	0.0960	0 0692	0.0419	0.0146	−0.0125	−0.0392	−0.0653
9	−0.0903	−0.1142	−0.1367	−0.1577	−0.1768	−0.1939	−0.2090	−0.2218	−0.2323	−0.2403
10	−0.2459	−0.2490	−0.2496	−0.2477	−0.2434	−0.2366	−0.2276	−0.2164	−0.2032	−0.1881
11	−0.1712	−0.1528	−0.1330	−0.1121	−0.0902	−0.0677	−0.0446	−0.0213	+0.0020	0.0250
12	0.0477	0.0697	0.0908	0.1108	0.1296	0.1469	0.1626	0.1766	0.1887	0.1988
13	0.2069	0.2129	0.2167	0.2183	0.2177	0.2150	0.2101	0.2032	0.1943	0.1836
14	0.1711	0.1570	0.1414	0.1245	0.1065	0.0875	0.0679	0.0476	0.0271	0 0064
15	−0.0142	−0.0346	−0.0544	−0.0736	−0.0919	−0.1092	−0.1253	−0.1401	−0.1533	−0.1650

* Reprinted with permission from *Reference Data for Radio Engineers*, published by the Federal Telephone and Radio Corporation, New York, 1943.

TABLE 2

$J_1(z)$

z	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.0000	0.0499	0.0995	0.1483	0.1960	0.2423	0.2867	0.3290	0.3688	0.4059
1	0.4401	0.4709	0.4983	0.5220	0.5419	0.5579	0.5699	0.5778	0.5815	0.5812
2	0.5767	0.5683	0.5560	0.5399	0.5202	0.4971	0.4708	0.4416	0.4097	0.3754
3	0.3391	0.3009	0.2613	0.2207	0.1792	0.1374	0.0955	0.0538	0.0128	−0.0272
4	−0.0660	−0.1033	−0.1386	−0.1719	−0.2028	−0.2311	−0.2566	−0.2791	−0.2985	−0.3147
5	−0.3276	−0.3371	−0.3432	−0.3460	−0.3453	−0.3414	−0.3343	−0.3241	−0.3110	−0.2951
6	−0.2767	−0.2559	−0.2329	−0.2081	−0.1816	−0.1538	−0.1250	−0.0953	−0.0652	−0.0349
7	−0.0047	+0.0252	0.0543	0.0826	0.1096	0.1352	0.1592	0.1813	0.2014	0.2192
8	0.2346	0.2476	0.2580	0.2657	0.2708	0.2731	0.2728	0.2697	0.2641	0.2559
9	0.2453	0.2324	0.2174	0.2004	0.1816	0.1613	0.1395	0.1166	0.0928	0.0684
10	0.0435	0.0184	−0.0066	−0.0313	−0.0555	−0.0789	−0.1012	−0.1224	−0.1422	−0.1603
11	−0.1768	−0.1913	−0.2039	−0.2143	−0.2225	−0.2284	−0.2320	−0.2333	−0.2323	−0.2290
12	−0.2234	−0.2157	−0.2060	−0.1943	−0.1807	−0.1655	−0.1487	−0.1307	−0.1114	−0.0912
13	−0.0703	−0.0489	−0.0271	−0.0052	+0.0166	0.0380	0.0590	0.0791	0.0984	0.1165
14	0.1334	0.1488	0.1626	0.1747	0.1850	0.1934	0.1999	0.2043	0.2066	0.2069
15	0.2051	0.2013	0.1955	0.1879	0.1784	0.1672	0.1544	0.1402	0.1247	0.1080

TABLE 3

$J_2(z)$

z	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.0000	0.0012	0.0050	0.0112	0.0197	0.0306	0.0437	0.0588	0.0758	0.0946
1	0.1149	0.1366	0.1593	0.1830	0.2074	0.2321	0.2570	0.2817	0.3061	0.3299
2	0.3528	0.3746	0.3951	0.4139	0.4310	0.4461	0.4590	0.4696	0.4777	0.4832
3	0.4861	0.4862	0.4835	0.4780	0.4697	0.4586	0.4448	0.4283	0.4093	0.3879
4	0.3641	0.3383	0.3105	0.2811	0.2501	0.2178	0.1846	0.1506	0.1161	0.0813

TABLE 4

$J_3(z)$

0	0.0000	0.0000	0.0002	0.0006	0.0013	0.0026	0.0044	0.0069	0.0102	0.0144
1	0.0196	0.0257	0.0329	0.0411	0.0505	0.0610	0.0725	0.0851	0.0988	0.1134
2	0.1289	0.1453	0.1623	0.1800	0.1981	0.2166	0.2353	0.2540	0.2727	0.2911
3	0.3091	0.3264	0.3431	0.3588	0.3734	0.3868	0.3988	0.4092	0.4180	0.4250
4	0.4302	0.4333	0.4344	0.4333	0.4301	0.4247	0.4171	0.4072	0.3952	0.3811

TABLE 5

$J_4(z)$

0	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003	0.0006	0.0010	0.0016
1	0.0025	0.0036	0.0050	0.0068	0.0091	0.0118	0.0150	0.0188	0.0232	0.0283
2	0.0340	0.0405	0.0476	0.0556	0.0643	0.0738	0.0840	0.0950	0.1067	0.1190
3	0.1320	0.1456	0.1597	0.1743	0.1891	0.2044	0.2198	0.2353	0.2507	0.2661
4	0.2811	0.2958	0.3100	0.3236	0.3365	0.3484	0.3594	0.3693	0.3780	0.3853

BESSEL FUNCTIONS—*Continued*

TABLE 6

p	$J_p(1)$	$J_p(2)$	$J_p(3)$	$J_p(4)$	$J_p(5)$	$J_p(6)$	$J_p(7)$	$J_p(8)$	$J_p(9)$	$J_p(10)$	$J_p(11)$	$J_p(12)$	$J_p(13)$	$J_p(14)$
0	+.7652	+.2239	−.2601	−.3971	−.1776	+.1506	+.3001	+.1717	−.09033	−.2459	−.1712	+.04769	+.2069	+.1711
0.5	+.6714	+.5130	+.06501	−.3019	−.3422	−.09102	+.1981	+.2791	+.1096	−.1373	−.2406	−.1236	+.09298	+.2112
1.0	+.4401	+.5767	+.3391	−.06604	−.3276	−.2767	$-.0^{2}4683$	+.2346	+.2453	+.04347	−.1768	−.2234	−.07032	+.1334
1.5	+.2403	+.4913	+.4777	+.1853	−.1697	−.3279	−.1991	+.07593	+.2545	+.1980	−.02293	−.2047	−.1937	−.01407
2.0	+.1149	+.3528	+.4861	+.3641	+.04657	−.2429	−.3014	−.1130	+.1448	+.2546	+.1390	−.08493	−.2177	−.1520
2.5	+.04950	+.2239	+.4127	+.4409	+.2404	−.07295	−.2834	−.2506	−.02477	+.1967	+.2343	+.07242	−.1377	−.2143
3.0	+.01956	+.1289	+.3091	+.4302	+.3648	+.1148	−.1676	−.2911	−.1809	+.05838	+.2273	+.1951	$+.0^{2}3320$	−.1768
3.5	$+.0^{2}7186$	+.06852	+.2101	+.3658	+.4100	+.2671	$-.0^{2}3403$	−.2326	−.2683	−.09965	+.1294	+.2348	+.1407	−.06245
4.0	$+.0^{2}2477$	+.03400	+.1320	+.2811	+.3912	+.3576	+.1578	−.1054	−.2655	−.2196	−.01504	+.1825	+.2193	+.07624
4.5	$+.0^{3}807$	+.01589	+.07760	+.1993	+.3337	+.3846	+.2800	+.04712	−.1839	−.2664	−.1519	+.06457	+.2134	+.1830
5.0	$+.0^{3}2498$	$+.0^{2}7040$	+.04303	+.1321	+.2611	+.3621	+.3479	+.1858	−.05504	−.2341	−.2383	−.07347	+.1316	+.2204
5.5	$+.0^{4}74$	$+.0^{2}2973$	+.02266	+.08261	+.1906	+.3098	+.3634	+.2856	+.08439	−.1401	−.2538	−.1864	$+.0^{2}7055$	+.1801
6.0	$+.0^{4}2094$	$+.0^{2}1202$	+.01139	+.04909	+.1310	+.2458	+.3392	+.3376	+.2043	−.01446	−.2016	−.2437	−.1180	+.08117
6.5	$+.0^{5}6$	$+.0^{3}467$	$+.0^{2}5493$	+.02787	+.08558	+.1833	+.2911	+.3456	+.2870	+.1123	−.1018	−.2354	−.2075	−.04151
7.0	$+.0^{5}1502$	$+.0^{3}1749$	$+.0^{2}2547$	+.01518	+.05338	+.1296	+.2336	+.3206	+.3275	+.2167	+.01838	−.1703	−.2406	−.1508
7.5	—	—	—	—	—	+.08741	+.1772	+.2759	+.3302	+.2861	+.1334	−.06865	−.2145	−.2187
8.0	$+.0^{7}9422$	$+.0^{4}2218$	$+.0^{3}4934$	$+.0^{2}4029$	+.01841	+.05653	+.1280	+.2235	+.3051	+.3179	+.2250	+.04510	−.1410	−.2320
8.5	—	—	—	—	—	+.03520	+.08854	+.1718	+.2633	+.3169	+.2838	+.1496	−.04006	−.1928
9.0	$+.0^{8}5249$	$+.0^{5}2492$	$+.0^{4}8440$	$+.0^{3}9386$	$+.0^{2}5520$	+.02117	+.05892	+.1263	+.2149	+.2919	+.3089	+.2304	+.06698	−.1143
9.5	—	—	—	—	—	+.01232	+.03785	+.08921	+.1672	+.2526	+.3051	+.2806	+.1621	−.01541
10.0	$+.0^{9}2631$	$+.0^{6}2515$	$+.0^{4}1293$	$+.0^{3}1950$	$+.0^{2}1468$	$+.0^{2}6964$	+.02354	+.06077	+.1247	+.2075	+.2804	+.3005	+.2338	+.08501

Note: $.0^{2}7186 = .007186$ $\qquad .0^{3}807 = .000807$

INDEX